HYPER VERSION

Received: *April 17, 2001,* Accepted: *May 7, 2001*

Stable wrapped branes

Jose M. Camino, Angel Paredes and Alfonso V. Ramallo

Departamento de Física de Partículas, Universidad de Santiago
E-15706 Santiago de Compostela, Spain
E-mail: camino@fpaxp1.usc.es, angel@fpaxp1.usc.es, alfonso@fpaxp1.usc.es

Abstract: We study some wrapped configurations of branes in the near-horizon geometry of a stack of other branes. The common feature of all the cases analyzed is a quantization rule and the appearance of a finite number of static configurations in which the branes are partially wrapped on spheres. The energy of these configurations can be given in closed form and the analysis of their small oscillations shows that they are stable. The cases studied include D$(8-p)$-branes in the type-II supergravity background of Dp-branes for $0 \leq p \leq 5$, M5-branes in the M5-brane geometry in M-theory and D3-branes in a (p,q) fivebrane background in the type-IIB theory. The brane configurations found admit the interpretation of bound states of strings (or M2-branes in M-theory) which extend along the unwrapped directions. We check this fact directly in a particular case by using the Myers polarization mechanism.

Keywords: M-Theory, D-branes, AdS-CFT Correspondance.

JHEP05(2001)011

Contents

1. Introduction

In a recent paper [1], Bachas, Douglas and Schweigert have shown how D-branes on a group manifold are stabilized against shrinking (see also ref. [2]). The concrete model studied in ref. [1] was the motion of a D2-brane in the geometry of the SU(2) group manifold. Topologically, SU(2) is equivalent to a three-sphere S^3. The D2-brane is embedded in this S^3 along a two-sphere S^2 which, in a system of spherical coordinates, is placed at constant latitude angle θ. The D2-brane dynamics is determined by the Born-Infeld action, in which a worldvolume gauge field is switched on. An essential ingredient in the analysis of ref. [1] is the quantization condition of the worldvolume flux, which, with our notations, can be written as:

$$\int_{S^2} F = \frac{2\pi n}{T_f}, \qquad n \in \mathbb{Z}, \tag{1.1}$$

where F is the worldvolume gauge field strength and T_f is the tension of the fundamental string, which, in terms of the Regge slope α' is $T_f = (2\pi\alpha')^{-1}$.

By using eq. (1.1) one can easily find the form of the worldvolume gauge field strength for non-zero n, and the corresponding value of the energy of the D2-brane. The minimum of this energy determines the embedding of the brane in the group manifold, which occurs at a finite set of latitude angles θ. It turns out that the static configurations found by this method are stable under small perturbations and exactly match those obtained by considering open strings on group manifolds [3, 4]. In this latter approach the D-brane configurations are determined by all the possible boundary conditions of the corresponding Conformal Field Theory (CFT). Actually [5, 6], each possible boundary condition corresponds to a D-brane wrapped on a (twisted) conjugacy class of the group.

The underlying CFT imposes quantization conditions on the allowed (twisted) conjugacy classes, which can be interpreted geometrically in terms of the embedding of the D-brane worldvolume in the group manifold. Thus, for example, in the case of the SU(2) group manifold, the non-trivial conjugacy classes are two-spheres embedded in SU(2) $\approx S^3$. The quantization conditions of the corresponding Wess-Zumino-Witten (WZW) model determine that only a finite number of $S^2 \subset S^3$ embeddings are possible and their number is related to the level of the affine su(2) Kac-Moody algebra [5, 6]. Actually, to each conjugacy class one associates a Cardy boundary state [7] of the WZW model. The mass of the D-brane configuration can be obtained, in this approach, by computing à la Polchinski [8] the matrix element between Cardy states of the string theory cylinder diagram, and comparing the result with the one obtained in a gravitational field theory. Apart from a finite shift in the level of the current algebra, the mass obtained in this way is exactly the same as the one computed with the Born-Infeld action and the quantization condition (1.1). For other aspects of this open string approach and of the flux quantization condition (1.1) see ref. [9].

The agreement between the Born-Infeld and CFT approaches for the system of ref. [1] is quite remarkable. For this reason the generalization of this result to other backgrounds and brane probes is very interesting. The SU(2) group manifold studied in [1] can be regarded as a component of the transverse part of a Neveu-Schwarz (NS) fivebrane geometry. Thus, the natural generalization to consider is a Ramond-Ramond (RR) background. This case was studied in ref. [10], where it was shown that the brane probe must be partially wrapped on some angular directions and extended along the radial coordinate.

Following the analysis of ref. [10], in this paper we study, first of all, the motion of a D$(8-p)$-brane in the background of a stack of parallel Dp-branes. The external region of the Dp-brane metric has SO$(9-p)$ rotational symmetry, which is manifest when a system of spherical coordinates is chosen. In this system of coordinates a transverse S^{8-p} sphere is naturally defined and the constant latitude condition on the S^{8-p} determines a S^{7-p} sphere. We shall embed the D$(8-p)$-brane in this background in such a way that it is wrapped on this $S^{7-p} \subset S^{8-p}$ constant latitude sphere and extended along the radial direction. Therefore, as in ref. [1], the brane configuration is characterized by an angle θ, which parametrizes the latitude of the S^{7-p}.

In order to analyze this Dp-D$(8-p)$ system by means of the Born-Infeld action, we shall establish first some quantization condition which, contrary to (1.1), will now involve the electric components of the worldvolume gauge field. By using this quantization rule we shall find a finite set of stable brane configurations characterized by some angles θ which generalize the ones found in ref. [1]. The energy of these configurations will be also computed and, from this result, we shall conclude that semiclassically our D$(8-p)$-brane configurations can be regarded as a bound state of fundamental strings. On the other hand, we will find a first order BPS differential equation whose fulfillment implies the saturation of an energy bound and whose constant θ solutions are precisely our wrapped configurations. This BPS equation is the one [11] satisfied by the baryon vertex [12], which will allow us to interpret our configurations as a kind of short distance limit (in the radial direction) of the baryonic branes [13]–[15].

Another purpose of this paper is to study a mechanism of flux stabilization in M-theory. We shall consider, in particular, a M5-brane probe in a M5-brane background. By using the Pasti-Sorokin-Tonin (PST) [16] action for the M5-brane probe, we shall look for static configurations in which the probe is wrapped on a three-sphere. After establishing a flux quantization condition similar to (1.1), we shall find these configurations and we will show that they closely resemble those found for the D4-D4 system. Actually, our states can be interpreted semiclassically as BPS bound states of M2-branes and they are related to the short distance limit of the baryonic vertex of M-theory [17, 18].

Another example which we will work out in detail is the one in which the background is a stack of fivebranes which have both NS and RR charges, i.e. a collection of the so-called (p, q) fivebranes [19]. In this case the probe is a D3-brane and the "magnetic" quantization condition (1.1) and our electric generalization must be imposed at the same time. We will show that these two quantization rules are indeed compatible and we will find the stable wrapped configurations of the D3-brane probe. Again, they can be interpreted semiclassically as a collection of strings (actually, in this case, (q, p) strings).

If our brane configurations admit an interpretation as bound states of strings (or M2-branes in the case of M-theory), it should be possible to obtain them starting directly from the strings (or M2-branes). We will check this fact in a particular case. Indeed, we will show how one can build up the wrapped D3-brane configurations in the NS fivebrane background by using D-strings in the same background. The mechanism responsible for this transmutation is the one advocated by Myers [20], in which the D-strings move in a noncommutative fuzzy sphere and are polarized by the background.

This paper is organized as follows. In section 2 we will study the Dp-D$(8-p)$ system. Section 3 is devoted to the analysis of the flux stabilization in M-theory. The D3-brane in the (p,q) fivebrane background is considered in section 4. In section 5 we summarize our results and explore some directions for future work. The paper is completed with two appendices. In appendix A we collect the functions which determine the location of the wrapped brane configurations in the transverse sphere. In appendix B we show how to obtain the wrapped D3-branes from polarized D-strings.

2. Wrapped branes in Ramond-Ramond backgrounds

The ten-dimensional metric corresponding to a stack of N coincident extremal Dp-branes in the near-horizon region is given by [21]:

$$ds^2 = \left[\frac{r}{R}\right]^{\frac{7-p}{2}} (-dt^2 + dx_{\|}^2) + \left[\frac{R}{r}\right]^{\frac{7-p}{2}} (dr^2 + r^2\, d\Omega_{8-p}^2)\,, \tag{2.1}$$

where $x_{\|}$ represent p cartesian coordinates along the branes, r is a radial coordinate parametrizing the distance to the branes and $d\Omega_{8-p}^2$ is the line element of an unit $8-p$ sphere. We have written the metric in the string frame. The parameter R, which we will refer to as the radius, is given by:

$$R^{7-p} = N g_s 2^{5-p} \pi^{\frac{5-p}{2}} (\alpha')^{\frac{7-p}{2}} \Gamma\left(\frac{7-p}{2}\right)\,, \tag{2.2}$$

where N is the number of Dp-branes of the stack and g_s is the string coupling constant. The metric (2.1) is a classical solution of the type-II supergravity equations of motion. This solution is also characterized by some non-vanishing values of the dilaton field $\phi(r)$ and of a Ramond-Ramond (RR) $(8-p)$-form field strength $F^{(8-p)}$, namely:

$$\begin{aligned} e^{-\tilde{\phi}(r)} &= \left[\frac{R}{r}\right]^{\frac{(7-p)(p-3)}{4}}\,, \\ F^{(8-p)} &= (7-p)\, R^{7-p}\, \epsilon_{(8-p)}\,, \end{aligned} \tag{2.3}$$

where $\tilde{\phi}(r) = \phi(r) - \phi(r\to\infty)$, and we are representing the Dp-brane as a magnetically charged object under the $F^{(8-p)}$ form. In eq. (2.3)(and in what follows) $\epsilon_{(n)}$ denotes the volume form of the sphere S^n.

Let θ^1, θ^2, ..., θ^{8-p} be coordinates which parametrize the S^{8-p} transverse sphere. We shall assume that the θ's are spherical angles on S^{8-p} and that $\theta \equiv \theta^{8-p}$ is the polar angle ($0 \le \theta \le \pi$). Therefore, the S^{8-p} line element $d\Omega_{8-p}^2$ can be decomposed as:

$$d\Omega_{8-p}^2 = d\theta^2 + (\sin\theta)^2 d\Omega_{7-p}^2\,. \tag{2.4}$$

In these coordinates it is not difficult to find a potential for the RR gauge field. Indeed, let us define the function $C_p(\theta)$ as the solution of the differential equation:

$$\frac{d}{d\theta}\,C_p(\theta) = -(7-p)\,(\sin\theta)^{7-p}\,, \tag{2.5}$$

with the initial condition

$$C_p(0) = 0\,. \tag{2.6}$$

It is clear that one can find by elementary integration a unique solution to the problem of eqs. (2.5) and (2.6). Thus $C_p(\theta)$ can be considered as a known function of the polar angle θ. In terms of $C_p(\theta)$, the RR potential $C^{(7-p)}$ can be represented as:

$$C^{(7-p)} = -R^{7-p}\,C_p(\theta)\epsilon_{(7-p)}\,. \tag{2.7}$$

By using eq. (2.5) it can be easily verified that:[1]

$$F^{(8-p)} = d\,C^{(7-p)}\,. \tag{2.8}$$

Let us now consider a D$(8-p)$-brane embedded along the transverse directions of the stack of Dp-branes. The action of such a brane probe is the sum of a Dirac-Born-Infeld and a Wess-Zumino term:

$$S = -T_{8-p}\int d^{9-p}\sigma\, e^{-\tilde\phi}\,\sqrt{-\det\,(g+F)} + T_{8-p}\int F\wedge C^{(7-p)}\,, \tag{2.9}$$

where g is the induced metric on the worldvolume of the D$(8-p)$-brane and F is a worldvolume abelian gauge field strength. The coefficient T_{8-p} in eq. (2.9) is the tension of the D$(8-p)$-brane, given by:

$$T_{8-p} = (2\pi)^{p-8}\,(\alpha')^{\frac{p-9}{2}}\,(g_s)^{-1}\,. \tag{2.10}$$

The worldvolume coordinates σ^α $(\alpha = 0,\ldots,8-p)$ will be taken as:

$$\sigma^\alpha = (t\,,\,r\,,\,\theta^1,\ldots,\theta^{7-p}\,)\,. \tag{2.11}$$

With this election the embedding of the brane probe is described by a function $\theta = \theta(\sigma^\alpha)$. Notice that the hypersurface θ = constant defines a S^{7-p} sphere on the transverse S^{8-p} (see figure 1). These configurations with constant polar angle represent a D$(8-p)$-brane wrapped on a S^{7-p} sphere and extended along the radial direction. These are the kind of configurations we want to study in this paper. Actually, we will consider first a more general situation in which the polar angle depends only on the radial coordinate, i.e. when $\theta = \theta(r)$. It is a rather simple exercise to compute the induced metric g in this case. Moreover, by inspecting the

[1] For simplicity, through this paper we choose the orientation of the transverse S^{8-p} sphere such that $\epsilon_{(8-p)} = (\sin\theta)^{7-p}\,d\theta\wedge\epsilon_{(7-p)}$.

JHEP05(2001)011

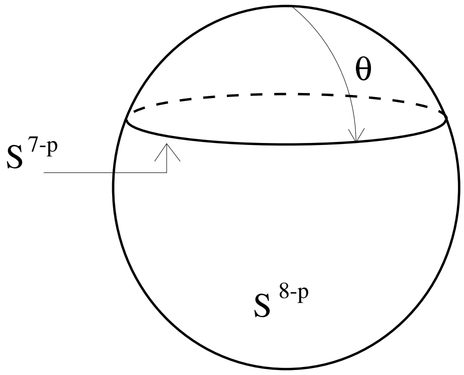

Figure 1: The points of the S^{8-p} sphere with the same polar angle θ define a S^{7-p} sphere. The angle θ represents the latitude on S^{8-p}, measured from one of its poles.

form of the RR potential $C^{(7-p)}$ in eq. (2.7) and the Wess-Zumino term in the action, one easily concludes that this term acts as a source for the worldvolume electric field $F_{0,r}$ and, thus, it is natural to assume that $F_{0,r}$ is different from zero. If we take this component of F as the only non-vanishing one, the action can be written as:

$$S = \int_{S^{7-p}} d^{7-p}\theta \,\sqrt{\hat{g}} \int drdt\, \mathcal{L}(\theta, F)\,, \tag{2.12}$$

where $\hat{g}$ is the determinant of the metric of the S^{7-p} and the lagrangian density $\mathcal{L}(\theta, F)$ is given by:

$$\mathcal{L}(\theta, F) = -\,T_{8-p}\, R^{7-p} \Big[(\sin\theta)^{7-p} \sqrt{1 + r^2\,\theta'^2 \;-\; F_{0,r}^2} + F_{0,r}\, C_p(\theta) \Big]\,. \tag{2.13}$$

In eq. (2.13) θ' denotes $d\theta/dr$.

2.1 Quantization condition

The equation of motion of the gauge field, derived from the lagrangian density of eq. (2.13), implies that:

$$\frac{\partial \mathcal{L}}{\partial F_{0,r}} = \text{constant}\,. \tag{2.14}$$

In order to determine the value of the constant on the right-hand side of eq. (2.14) let us follow the procedure of ref. [10] and couple the D-brane to a Neveu-Schwarz (NS) Kalb-Ramond field B. As is well-known, this coupling can be performed by substituting F by $F - B$ in $\mathcal{L}$, i.e. by doing $\mathcal{L}(\theta, F) \rightarrow \mathcal{L}(\theta, F - B)$ in eq. (2.13). At first order in B, this substitution generates a coupling of the D-brane to the NS field B of the form:

$$\int_{S^{7-p}} d^{7-p}\theta \,\sqrt{\hat{g}} \int drdt\, \frac{\partial\, \mathcal{L}}{\partial F_{0,r}}\, B_{0,r}\,, \tag{2.15}$$

where we have assumed that only the $B_{0,r}$ component of the B field is turned on.

We shall regard eq. (2.15) as the interaction energy of a fundamental string source in the presence of the D-brane. This source is extended along the radial direction and, thus, it is quite natural to require that the coefficient of the B field, integrated over S^{7-p}, be an integer multiple of the fundamental string tension, namely:

$$\int_{S^{7-p}} d^{7-p}\theta\,\sqrt{\hat{g}}\,\frac{\partial\,\mathcal{L}}{\partial F_{0,r}} = n\,T_f\,, \tag{2.16}$$

with $n \in \mathbb{Z}$. Eq. (2.16) is the quantization condition we were looking for in these RR backgrounds and will play in our analysis a role similar to the one played in ref. [1] by the flux quantization condition (eq. (1.1)). Notice that eq. (2.16) constraints the electric components of F, whereas eq. (1.1) involves the magnetic worldvolume field.[2] Thus, our quantization rule is a kind of electric-magnetic dual of the one used in ref. [1]. This has a nice interpretation in the case in which p is odd, which corresponds to the type-IIB theory. Indeed, it is known in this case that the electric-magnetic worldvolume duality corresponds to the S-duality of the background [22]. In particular, when $p = 5$, the D5 background can be converted, by means of an S-duality transformation, into a NS5 one, which is precisely the type of geometry considered in ref. [1].

By using the explicit form of the lagrangian density (eq. (2.13)), the left-hand side of our quantization condition can be easily calculated:

$$\int_{S^{7-p}} d^{7-p}\theta\,\sqrt{\hat{g}}\,\frac{\partial\,\mathcal{L}}{\partial F_{0,r}} = T_{8-p}\,\Omega_{7-p}\,R^{7-p}\left[\frac{F_{0,r}\,(\sin\theta)^{7-p}}{\sqrt{1+r^2\,\theta'^2 - F_{0,r}^2}} - C_p(\theta)\right]\,, \tag{2.17}$$

where Ω_{7-p} is the volume of the unit $(7-p)$-sphere, given by:

$$\Omega_{7-p} = \frac{2\pi^{\frac{8-p}{2}}}{\Gamma\left(\frac{8-p}{2}\right)}\,. \tag{2.18}$$

By using eqs. (2.17) and (2.16) one can obtain $F_{0,r}$ as a function of $\theta(r)$ and the integer n. Let us show how this can be done. First of all, by using eqs. (2.10), (2.18) and (2.2) it is straightforward to compute the global coefficient appearing on the right-hand side of eq. (2.17), namely:

$$T_{8-p}\,\Omega_{7-p}\,R^{7-p} = \frac{NT_f}{2\sqrt{\pi}}\,\frac{\Gamma\left(\frac{7-p}{2}\right)}{\Gamma\left(\frac{8-p}{2}\right)}\,. \tag{2.19}$$

Secondly, let us define the function $\mathcal{C}_{p,n}(\theta)$ as:

$$\mathcal{C}_{p,n}(\theta) = C_p(\theta) + 2\,\sqrt{\pi}\,\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\,\frac{n}{N}\,. \tag{2.20}$$

[2] It is interesting to point out that the left-hand side of eq. (2.16) can be written in terms of the integral over the S^{7-p} sphere of the worldvolume Hodge dual of $\partial\,\mathcal{L}/\partial F_{\alpha,\beta}$.

Notice that $\mathcal{C}_{p,n}(\theta)$ satisfies the same differential equation as $C_p(\theta)$ (eq. (2.5)) with different initial condition. Moreover, by inspecting eqs. (2.16), (2.17) and (2.19), one easily concludes that $F_{0,r}$ can be put in terms of $\mathcal{C}_{p,n}(\theta)$. The corresponding expression is:

$$F_{0,r} = \sqrt{\frac{1 + r^2\,\theta'^2}{\mathcal{C}_{p,n}(\theta)^2 + (\sin\theta)^{2(7-p)}}}\,\mathcal{C}_{p,n}(\theta)\,. \tag{2.21}$$

Let us now evaluate the energy of the system. By performing a Legendre transformation, we can represent the hamiltonian H of the D$(8-p)$-brane as:

$$H = \int_{S^{7-p}} d^{7-p}\theta\sqrt{\hat{g}}\,\int dr\,\left[F_{0,r}\,\frac{\partial\,\mathcal{L}}{\partial F_{0,r}} - \mathcal{L}\right]. \tag{2.22}$$

By using (2.21) one can eliminate $F_{0,r}$ from the expression of H. One gets:

$$H = T_{8-p}\,\Omega_{7-p}\,R^{7-p}\int dr\,\sqrt{1 + r^2\,\theta'^2}\sqrt{\mathcal{C}_{p,n}(\theta)^2 + (\sin\theta)^{2(7-p)}}\,. \tag{2.23}$$

It is now simple to find the constant θ configurations which minimize the energy. Indeed, we only have to require the vanishing of $\partial H/\partial\theta$ for $\theta' = 0$. Taking into account that $\mathcal{C}_{p,n}(\theta)$ satisfies eq. (2.5), we arrive at:

$$\frac{\partial H}{\partial\theta}\bigg|_{\theta'=0} = (7-p)\,T_{8-p}\,\Omega_{7-p}\,R^{7-p}\,\frac{(\sin\theta)^{7-p}[(\sin\theta)^{6-p}\cos\theta - \mathcal{C}_{p,n}(\theta)]}{\sqrt{\mathcal{C}_{p,n}(\theta)^2 + (\sin\theta)^{2(7-p)}}}\,. \tag{2.24}$$

Moreover, if we define the function:[3]

$$\Lambda_{p,n}(\theta) \equiv (\sin\theta)^{6-p}\,\cos\theta - \mathcal{C}_{p,n}(\theta)\,, \tag{2.25}$$

it is clear by looking at the right-hand side of eq. (2.24) that the energy is minimized either when $\theta = 0, \pi$ (i.e. when $\sin\theta = 0$) or when $\theta = \bar{\theta}_{p,n}$, where $\bar{\theta}_{p,n}$ is determined by the condition:

$$\Lambda_{p,n}(\bar{\theta}_{p,n}) = 0\,. \tag{2.26}$$

The solutions $\theta = 0, \pi$ correspond to singular configurations in which the D$(8-p)$-brane collapses at the poles of the S^{7-p} sphere. For this reason we shall concentrate on the analysis of the $\theta = \bar{\theta}_{p,n}$ configurations. First of all, we notice that the function $\Lambda_{p,n}(\theta)$ has a simple derivative, which can be obtained from its definition and from the differential equation satisfied by $\mathcal{C}_{p,n}(\theta)$. One gets:

$$\frac{d}{d\theta}\,\Lambda_{p,n}(\theta) = (6-p)\,(\sin\theta)^{5-p}\,. \tag{2.27}$$

It follows from eq. (2.27) that when $p < 6$ then $\frac{d}{d\theta}\,\Lambda_{p,n}(\theta) > 0$ if $\theta \in (0, \pi)$. This means that, for $p \leq 5$, $\Lambda_{p,n}(\theta)$ is a monotonically increasing function in the interval

[3]The functions $\Lambda_{p,n}(\theta)$ for different values of p have been listed in appendix A.

JHEP05(2001)011

$0 < \theta < \pi$. In what follows we shall restrict ourselves to the case $p \leq 5$. In order to check that eq. (2.26) has solutions in this case, let us evaluate the values of $\Lambda_{p,n}(\theta)$ at $\theta = 0, \pi$. From eqs. (2.25), (2.20) and (2.6) we have:

$$\Lambda_{p,n}(0) = -\mathcal{C}_{p,n}(0) = -2\sqrt{\pi}\,\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\,\frac{n}{N}\,. \tag{2.28}$$

Moreover for $\theta = \pi$ we can write:

$$\Lambda_{p,n}(\pi) = -\,\mathcal{C}_{p,n}(\pi) = -C_p(\pi) - 2\sqrt{\pi}\,\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\,\frac{n}{N}\,, \tag{2.29}$$

and, taking into account that:

$$C_p(\pi) = -(7-p)\int_0^{\pi}(\sin\theta)^{7-p}\,d\theta \;=\; -2\sqrt{\pi}\,\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\,, \tag{2.30}$$

we get:

$$\Lambda_{p,n}(\pi) = 2\sqrt{\pi}\,\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\,(1-\frac{n}{N})\,. \tag{2.31}$$

As $\frac{d}{d\theta}\,\Lambda_{p,n}(\theta) > 0$ for $\theta \in (0,\pi)$, the function $\Lambda_{p,n}(\theta)$ vanishes for $0 < \theta < \pi$ if and only if $\Lambda_{p,n}(0) < 0$ and $\Lambda_{p,n}(\pi) > 0$. From eq. (2.28) we conclude that the first condition occurs when $n > 0$, whereas eq. (2.31) shows that $\Lambda_{p,n}(\pi) > 0$ if $n < N$. It follows that there exists only one solution $\bar{\theta}_{p,n} \in (0,\pi)$ of eq. (2.26) for each n in the interval $0 < n < N$. Then, we have found exactly $N-1$ angles which correspond to nonsingular wrappings of the D$(8-p)$-brane on a S^{7-p} sphere. Notice that for $n = 0$ $(n = N)$ the solution of eq. (2.26) is $\bar{\theta}_{p,0} = 0$ $(\bar{\theta}_{p,N} = \pi)$ (see eqs. (2.28) and (2.31)). Therefore, we can identify these $n = 0, N$ cases with the singular configurations previously found. In general, when n is varied from $n = 0$ to $n = N$ the angle $\bar{\theta}_{p,n}$ increases from 0 to π (i.e. from one of the poles of the S^{8-p} sphere to the other).

It is not difficult to find the energy of these wrapped configurations. Actually we only need to substitute $\theta = \bar{\theta}_{p,n}$ in eq. (2.23). Taking into account (see eqs. (2.25) and (2.26)) that:

$$\mathcal{C}_{p,n}(\bar{\theta}_{p,n}) = (\sin\bar{\theta}_{p,n})^{6-p}\,\cos\bar{\theta}_{p,n}\,, \tag{2.32}$$

one easily finds that the energy of these solutions can be written as:

$$H_{p,n} = \int dr\,\mathcal{E}_{p,n}\,, \tag{2.33}$$

where the constant energy density $\mathcal{E}_{p,n}$ is given by:

$$\mathcal{E}_{p,n} = \frac{NT_f}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{7-p}{2}\right)}{\Gamma\left(\frac{8-p}{2}\right)} (\sin\bar{\theta}_{p,n})^{6-p} \,. \tag{2.34}$$

Similarly, by substituting eq. (2.32) in eq. (2.21), we can get the worldvolume electric field for our configurations, namely:

$$\bar{F}_{0,r} = \cos\bar{\theta}_{p,n} \,. \tag{2.35}$$

Let us now analyze some particular cases of our equations. First of all, we shall consider the $p = 5$ case, i.e. a D3-brane wrapped on a two-sphere under the action of a D5-brane background. The function $\Lambda_{5,n}(\theta)$ is:

$$\Lambda_{5,n}(\theta) = \theta - \frac{n}{N}\pi \,, \tag{2.36}$$

and the equation $\Lambda_{5,n}(\theta) = 0$ is trivially solved by the angles:

$$\bar{\theta}_{5,n} = \frac{n}{N}\pi \,. \tag{2.37}$$

Notice that the set of angles in eq. (2.37) is the same as that of ref. [1]. Using this result in eq. (2.34) we get the following energy density:

$$\mathcal{E}_{5,n} = \frac{NT_f}{\pi} \sin\left[\frac{n}{N}\pi\right] , \tag{2.38}$$

which is very similar to the result found in ref. [1]. Next, let us take $p = 4$, which corresponds to a D4-brane wrapped on a three-sphere in the background of a stack of D4-branes. The corresponding $\Lambda_{p,n}(\theta)$ function is:

$$\Lambda_{4,n}(\theta) = -2\left[\cos\theta + 2\,\frac{n}{N} - 1\right] , \tag{2.39}$$

and the solutions of eq. (2.26) in this case are easily found, namely:

$$\cos\bar{\theta}_{4,n} = 1 - 2\,\frac{n}{N} \,. \tag{2.40}$$

The corresponding energy density takes the form:

$$\mathcal{E}_{4,n} = \frac{n(N-n)}{N}\, T_f \,. \tag{2.41}$$

Notice that, in this D4-D4 case, the energy density of eq. (2.41) is a rational fraction of the fundamental string tension.

For general p the equation $\Lambda_{p,n}(\theta) = 0$ is much difficult to solve analytically. In order to illustrate this point let us write down the equation to solve in the physically interesting case $p = 3$:

$$\bar{\theta}_{3,n} - \cos\bar{\theta}_{3,n}\,\sin\bar{\theta}_{3,n} = \frac{n}{N}\pi \,. \tag{2.42}$$

Despite of the fact that we are not able to find the analytical solution of the equation $\Lambda_{p,n}(\theta) = 0$ for $p \leq 3$, we can get some insight on the nature of our solutions from some general considerations. First of all, it is interesting to point out the following property of the functions $\Lambda_{p,n}(\theta)$:

$$\Lambda_{p,n}(\theta) = -\Lambda_{p,N-n}(\pi - \theta)\,. \tag{2.43}$$

Eq. (2.43) can be proved either from the definition of the $\Lambda_{p,n}(\theta)$'s or from their expressions listed in appendix A. It follows from this equation that our set of angles $\bar{\theta}_{p,n}$ satisfy:

$$\bar{\theta}_{p,N-n} = \pi - \bar{\theta}_{p,n}\,. \tag{2.44}$$

By using (2.44) in the expression of the energy density $\mathcal{E}_{p,n}$ (eq. (2.34)), one immediately gets the following periodicity relation:

$$\mathcal{E}_{p,N-n} = \mathcal{E}_{p,n}\,. \tag{2.45}$$

Another interesting piece of information can be obtained by considering the semiclassical $N \to \infty$ limit. Notice that $\Lambda_{p,n}$ depends on n and N through their ratio n/N (see eqs. (2.25)) and (2.20)). Then, taking $N \to \infty$ with fixed n is equivalent to make $n \to 0$ for finite N. We have already argued that if $n \to 0$ the angle $\bar{\theta}_{p,n} \to 0$. In order to solve the equation $\Lambda_{p,n}(\theta) = 0$ for small θ, let us expand $\Lambda_{p,n}(\theta)$ in Taylor series around $\theta = 0$. It turns out [15] that the first non-vanishing derivative of $\Lambda_{p,n}(\theta)$ at $\theta = 0$ is the $(6-p)^{th}$ one and, actually, near $\theta = 0$, we can write:

$$\Lambda_{p,n}(\theta) \approx \Lambda_{p,n}(0) + \theta^{6-p} + \cdots\,. \tag{2.46}$$

It follows immediately that for $N \to \infty$ the value of $\bar{\theta}_{p,n}$ is given by:

$$(\bar{\theta}_{p,n})^{6-p} \approx -\Lambda_{p,n}(0)\,. \tag{2.47}$$

Taking into account eq. (2.28) and the general expression of the energy density (eq. (2.34)), we can easily verify that:

$$\lim_{N\to\infty} \mathcal{E}_{p,n} = n\,T_f\,, \tag{2.48}$$

a fact which can be verified directly for $p = 4, 5$ from our analytical expressions of the energy density (eqs. (2.38) and (2.41)). It is now clear from eq. (2.48) that our configurations can be interpreted as bound states of n fundamental strings. Actually, one can prove quite generally that the following inequality holds:

$$\mathcal{E}_{p,n} \leq n\,T_f\,, \tag{2.49}$$

which shows that the formation of our bound states is energetically favored. This is an indication of their stability, which we will verify directly in section 2.3.

In order to prove (2.49), it is very useful again to consider the dependence of the energy on $1/N$. Notice that for $1/N \to 0$ both sides of eq. (2.49) are equal (see eq. (2.48)). The energy $\mathcal{E}_{p,n}$ depends on $1/N$ both explicitly and implicitly (through $\bar{\theta}_{p,n}$). If we consider $1/N$ as a continuous variable, then one has:

$$\frac{d}{d(1/N)}\,\bar{\theta}_{p,n} = \frac{nNT_f}{(6-p)\,\mathcal{E}_{p,n}}\,\sin\bar{\theta}_{p,n}\,. \tag{2.50}$$

Eq. (2.50) is obtained by differentiating eq. (2.26) and using eqs. (2.27) and (2.28) (the latter determines the explicit dependence of $\Lambda_{p,n}$ on $1/N$). We are now ready to demonstrate (2.49). For this purpose let us consider the quantity $(\mathcal{E}_{p,n} - n\,T_f)/N$, which we will regard as a function of $1/N$. We must prove that this quantity is always less or equal than zero. Clearly, eq. (2.48) implies that $(\mathcal{E}_{p,n} - n\,T_f)/N \to 0$ for $1/N \to 0$. Moreover, by using (2.50) it is straightforward to compute the derivative:

$$\frac{d}{d(1/N)}\left[\frac{\mathcal{E}_{p,n} - n\,T_f}{N}\right] = -n\,T_f(1-\cos\bar{\theta}_{p,n})\,, \tag{2.51}$$

which vanishes for $N \to \infty$ and is always negative for finite N and $0 < n < N$. Thus, it follows that $(\mathcal{E}_{p,n} - n\,T_f)/N$ is negative for finite N and, necessarily, eq. (2.49) holds.

As a further check of (2.49) one can compute the first correction to $\mathcal{E}_{p,n} - n\,T_f$ for finite N. By Taylor expanding $\mathcal{E}_{p,n}$ in powers of $1/N$, and using eq. (2.50), one can prove that:

$$\mathcal{E}_{p,n}\ -\ n\,T_f \approx -\frac{6-p}{2(8-p)}\,n\,T_f\Big(\mathcal{C}_{p,n}(0)\Big)^{\frac{2}{6-p}}\ +\cdots\,, \tag{2.52}$$

where $\mathcal{C}_{p,n}(0)$, which is of order $1/N$, has been given in eq. (2.28).

2.2 BPS configurations and the baryon vertex

In this section we shall show that the wrapped configurations found above solve a BPS differential equation. With this purpose in mind, let us now come back to the more general situation in which the angle θ depends on the radial coordinate r. The hamiltonian for a general function $\theta(r)$ was given in eq. (2.23). By means of a simple calculation it can be verified that this hamiltonian can be written as:

$$H = T_{8-p}\,\Omega_{7-p}\,R^{7-p}\int dr\,\sqrt{\mathcal{Z}^2 + \mathcal{Y}^2}\,, \tag{2.53}$$

where, for any function $\theta(r)$, $\mathcal{Z}$ is a total derivative:

$$\mathcal{Z} = \frac{d}{dr}\Big[r\Big((\sin\theta)^{6-p} - \Lambda_{p,n}(\theta)\cos\theta\Big)\Big]\,, \tag{2.54}$$

and $\mathcal{Y}$ is given by:

$$\mathcal{Y} = \sin\theta\,\Lambda_{p,n}(\theta) - r\theta'\Big[(\sin\theta)^{6-p} - \Lambda_{p,n}(\theta)\cos\theta\Big]\,. \tag{2.55}$$

It follows from eq. (2.53) that H is bounded as:

$$H \geq T_{8-p}\,\Omega_{7-p}\,R^{7-p}\int dr\,|\,\mathcal{Z}\,|\,. \tag{2.56}$$

Since $\mathcal{Z}$ is a total derivative, the bound on the right-hand side of eq. (2.56) only depends on the boundary values of $\theta(r)$. This implies that any $\theta(r)$ saturating the bound is also a solution of the equations of motion. This saturation of the bound clearly occurs when $\mathcal{Y}=0$ or, taking into account eq. (2.55), when $\theta(r)$ satisfies the following first-order differential equation:

$$\theta\,' = \frac{1}{r}\,\frac{\sin\theta\,\Lambda_{p,n}(\theta)}{(\sin\theta)^{6-p}-\Lambda_{p,n}(\theta)\,\cos\theta}\,. \tag{2.57}$$

It is straightforward to verify directly that any solution $\theta(r)$ of eq. (2.57) also solves the second-order differential equations of motion derived from the hamiltonian of eq. (2.23). Moreover, by using eq. (2.57) to evaluate the right-hand side of eq. (2.21), one can demonstrate that the BPS differential equation is equivalent to the following relation between the electric field $F_{0,r}$ and $\theta(r)$:

$$F_{0,r} = \partial_r\,(r\cos\theta) = \cos\theta - r\theta\,'\,\sin\theta\,. \tag{2.58}$$

Notice now that eq. (2.57) admits solutions with $\theta =$ constant if and only if $\theta = 0\,,\,\pi$ or when θ is a zero of $\Lambda_{p,n}(\theta)$. Thus our wrapped configurations are certainly solutions of the BPS differential equation. As a confirmation of this fact, let us point out that, for constant θ, the electric field of eq. (2.58) reduces to the value displayed in eq. (2.35).

Eq. (2.57) was first proposed (for $p=3$) in ref. [11] to describe the baryon vertex (see also refs. [13]–[15]).[4] In ref. [18] it was verified, by looking at the κ-symmetry of the brane probe, that the condition (2.58) is enough to preserve 1/4 of the bulk supersymmetry. Actually, following the results of ref. [15], it is not difficult to obtain the general solution of the BPS differential equation (2.57). In implicit form this solution can be written as:

$$\frac{[\Lambda_{p,n}(\theta)]^{\frac{1}{6-p}}}{\sin\theta} = C\,r\,, \tag{2.59}$$

where C is a constant. Our constant angle solutions $\theta = \bar\theta_{p,n}$ can be obtained from eq. (2.59) by taking $C=0$, whereas the baryon vertex solutions correspond to $C\neq 0$. A glance at eq. (2.59) reveals that, by consistency, θ must be restricted to take values in an interval such that the function $\Lambda_{p,n}(\theta)$ has a fixed sign. If, for example, $\theta\in(0,\bar\theta_{p,n})$, then $\Lambda_{p,n}(\theta)<0$ and, by redefining the phase of C, we get

[4] In these studies of the baryon vertex a different choice of worldvolume coordinates is performed. Instead of taking these coordinates as in eq. (2.11), one takes $\sigma^{\alpha} = (t\,,\theta^1\,,\ldots\,,\theta^{7-p}\,,\theta)$ and the embedding of the D$(8-p)$-brane is described by a function $r = r(\sigma^{\alpha})$.

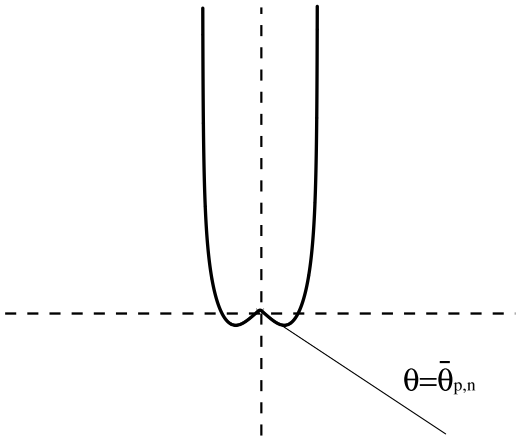

Figure 2: Representation of a typical solution of the BPS equation (2.59) for $C \neq 0$. In this plot r and θ are the polar coordinates of the plane of the figure. We have also plotted the $\theta = \bar{\theta}_{p,n}$ curve, which is the solution of (2.59) for $C = 0$.

a consistent solution in which r is a non-negative real number. Similarly, we could have $\theta \in (\bar{\theta}_{p,n}, \pi)$ since $\Lambda_{p,n}(\theta) > 0$ for these values. In both cases $\bar{\theta}_{p,n}$ is a limiting angle. Actually, for $0 < n < N$ one immediately infers from eq. (2.59) that $\bar{\theta}_{p,n}$ is the angle reached when $r \to 0$. The baryon vertex solutions behave [13]–[15] as a bundle of fundamental strings in the asymptotic region $r \to \infty$ (see figure 2). The number of fundamental strings is precisely n for the solution with $\theta \in (0, \bar{\theta}_{p,n})$ (and $N - n$ when $\theta \in (\bar{\theta}_{p,n}, \pi)$). Notice that $r \to \infty$ when $\theta = 0$ ($\theta = \pi$) for the solution with n ($N - n$) fundamental strings, whereas in the opposite limit $r \to 0$ the solution displayed in eq. (2.59) is equivalent to our $\theta = \bar{\theta}_{p,n}$ wrapped configuration. This is quite suggestive and implies that one can regard our constant angle configurations as a short distance limit (in the radial direction) of the baryon vertex solutions.

2.3 Fluctuations and stability

We are now going to study fluctuations around the static configurations found above. Let us parametrize these fluctuations as follows:

$$\theta = \bar{\theta}_{p,n} + \xi \, , \qquad F_{0,r} = \cos \bar{\theta}_{p,n} + f \, , \tag{2.60}$$

where ξ and f are small quantities which depend on the worldvolume coordinates σ^{α}. We are going to prove in this section that the $\theta = \bar{\theta}_{p,n}$ solution is stable under the perturbation of eq. (2.60). In order to achieve this goal we must go back to the action written in eq. (2.9). We shall evaluate this action for an angle θ and an electric field as in eq. (2.60). Let us represent the perturbation f by means of a potential as $f = \partial_0 a_r - \partial_r a_0$. We shall choose a gauge in which the components $a_{\hat{i}}$ of the potential along the sphere S^{7-p} vanish. Then we see that, for consistency, we must include in our perturbation the components of the gauge field strength of the type $F_{\hat{i},r} = \partial_{\hat{i}} a_r$ and $F_{0,\hat{i}} = -\partial_{\hat{i}} a_0$. Under these circumstances it is not difficult to compute the lagrangian density for the action (2.9) up to second order in ξ, f, $F_{\hat{i},r}$ and $F_{0,\hat{i}}$. After some calculation one gets:

$$\begin{aligned}\mathcal{L} = & -R^{7-p}\, T_{8-p}\, \Lambda_{p,n}(0)\, f + R^{7-p}\, T_{8-p}\, (\sin\bar{\theta})^{6-p} \times \\ & \times \frac{1}{2}\Bigg\{ R^{7-p}\, r^{p-5}\, (\partial_0\xi)^2 - r^2\,(\partial_r\xi)^2 - (\partial_{\hat{i}}\xi)^2 + \frac{R^{p-7}\, r^{5-p}}{(\sin\bar{\theta})^2} \times \\ & \times \left[\left(\frac{R}{r}\right)^{7-p} F_{0,\hat{i}}^2 - F_{\hat{i},r}^2\right] + (7-p)\xi^2 + \frac{f^2}{(\sin\bar{\theta})^2} + 2(7-p)\,\frac{f\,\xi}{\sin\bar{\theta}}\Bigg\}\,,\end{aligned} \tag{2.61}$$

where, to simplify the notation, we have written $\bar{\theta}$ instead of $\bar{\theta}_{p,n}$, $\hat{g}_{\hat{i}\hat{j}}$ represents the metric of the S^{7-p} sphere and we have denoted:

$$(\partial_{\hat{i}}\xi)^2 = \hat{g}^{\hat{i}\hat{j}}\partial_{\hat{i}}\xi\,\partial_{\hat{j}}\xi\,, \qquad F_{\hat{i},r}^2 = \hat{g}^{\hat{i}\hat{j}} F_{\hat{i},r}\, F_{\hat{j},r}\,, \qquad F_{0,\hat{i}}^2 = \hat{g}^{\hat{i}\hat{j}} F_{0,\hat{i}}\, F_{0,\hat{j}}\,. \tag{2.62}$$

In eq. (2.61) we have dropped the zero-order term. Moreover, the first term on the right-hand side of eq. (2.61) is a first-order term which, however, does not contribute to the equations of motion. In fact, by computing the variation of the action with respect to a_0, a_r and ξ we get the following set of equations:

$$\begin{aligned}\partial_r\left[\frac{f}{\sin\bar{\theta}} + (7-p)\,\xi\right] + \frac{1}{r^2\sqrt{\hat{g}}}\,\partial_{\hat{i}}\left[\sqrt{\hat{g}}\,\hat{g}^{\hat{i}\hat{j}}\,\frac{F_{0,\hat{j}}}{\sin\bar{\theta}}\right] &= 0\,,\\ r^{p-5}\,R^{7-p}\partial_0\left[\frac{f}{\sin\bar{\theta}} + (7-p)\,\xi\right] - \frac{1}{\sqrt{\hat{g}}}\,\partial_{\hat{i}}\left[\sqrt{\hat{g}}\,\hat{g}^{\hat{i}\hat{j}}\,\frac{F_{\hat{j},r}}{\sin\bar{\theta}}\right] &= 0\,,\\ R^{7-p}r^{p-5}\,\partial_0^2\xi - \partial_r(r^2\partial_r\xi) - \nabla^2_{S^{(7-p)}}\,\xi + (p-7)\left[\xi + \frac{f}{\sin\bar{\theta}}\right] &= 0\,.\end{aligned} \tag{2.63}$$

The first equation in (2.63) is nothing but the Gauss law. Moreover, if we further fix the gauge to $a_0 = 0$ (i.e. $f = \partial_0 a_r$, $F_{\hat{i},r} = \partial_{\hat{i}} a_r$ and $F_{0,\hat{i}} = 0$), the second equation in (2.63) can be written as:

$$r^{p-5}\,R^{7-p}\left[\frac{\partial_0^2 a_r}{\sin\bar{\theta}} + (7-p)\,\partial_0\xi\right] - \frac{1}{\sin\bar{\theta}}\,\nabla^2_{S^{(7-p)}}\,a_r = 0\,, \tag{2.64}$$

where $\nabla^2_{S^{(7-p)}}$ is the laplacian operator on the $S^{(7-p)}$ sphere. In order to continue with our analysis, let us now expand a_r and ξ in spherical harmonics of $S^{(7-p)}$:

$$\begin{aligned}a_r(t,r,\theta^1,\ldots,\theta^{7-p}) &= \sum_{l\geq 0,m} Y_{l,m}(\theta^1,\ldots,\theta^{7-p})\,\alpha_{l,m}(t,r)\,,\\ \xi(t,r,\theta^1,\ldots,\theta^{7-p}) &= \sum_{l\geq 0,m} Y_{l,m}(\theta^1,\ldots,\theta^{7-p})\,\zeta_{l,m}(t,r)\,.\end{aligned} \tag{2.65}$$

The spherical harmonics $Y_{l,m}$ are well-defined functions on $S^{(7-p)}$ which are eigenfunctions of the laplacian on the sphere, namely:

$$\nabla^2_{S^{(7-p)}} Y_{l,m} = -l(l+6-p)\,Y_{l,m}\,. \tag{2.66}$$

By plugging the mode expansion (2.65) into the equations of motion (2.63) and (2.64), and using eq. (2.66), we can obtain some equations for $\alpha_{l,m}(t,r)$ and $\zeta_{l,m}(t,r)$. Actually, if we define:

$$\eta_{l,m} \equiv \frac{\partial_0 \alpha_{l,m}}{\sin\bar{\theta}} + (7-p)\,\zeta_{l,m}\,, \tag{2.67}$$

then, the Gauss law in this $a_0 = a_{\hat{i}} = 0$ gauge can be simply written as:

$$\partial_r\,\eta_{l,m} = 0\,, \tag{2.68}$$

whereas the other two equations of motion give rise to:

$$\begin{aligned} R^{7-p}\,r^{p-5}\,\partial_0\left[\frac{\partial_0 \alpha_{l,m}}{\sin\bar{\theta}} + (7-p)\,\zeta_{l,m}\right] + l(l+6-p)\,\frac{\alpha_{l,m}}{\sin\bar{\theta}} = 0\,, \\ R^{7-p}\,r^{p-5}\partial_0^2\zeta_{l,m} - \partial_r(r^2\partial_r\zeta_{l,m}) + l(l+6-p)\zeta_{l,m} + (p-7)\left[\zeta_{l,m} + \frac{\partial_0\alpha_{l,m}}{\sin\bar{\theta}}\right] = 0. \end{aligned} \tag{2.69}$$

Let us analyze first eqs. (2.68) and (2.69) for $l = 0$. From the first equation in (2.69) it follows that:

$$\partial_0\eta_{0,m} = 0\,. \tag{2.70}$$

Thus, as $\partial_r\eta_{0,m} = 0$ (see eq. (2.68)), one concludes that:

$$\eta_{0,m} = \text{constant}\,. \tag{2.71}$$

By using this result and the definition of $\eta_{l,m}$ given in eq. (2.67), we can express $\partial_0\,\alpha_{0,m}$ in terms of $\zeta_{0,m}$ and the additive constant appearing in eq. (2.71). By substituting this relation in the second equation in (2.69), we get:

$$R^{7-p}\,r^{p-5}\,\partial_0^2\zeta_{0,m} - \partial_r(r^2\,\partial_r\,\zeta_{0,m}) + (6-p)(7-p)\,\zeta_{0,m} = \text{constant}\,. \tag{2.72}$$

It is interesting to rewrite eq. (2.72) in the following form. First of all, we define the wave operator $\mathcal{O}_p$ that acts on any function ψ as:

$$\mathcal{O}_p\,\psi \equiv R^{7-p}\,r^{p-5}\,\partial_0^2\,\psi - \partial_r\,(r^2\,\partial_r\psi)\,. \tag{2.73}$$

Then, if m_0^2 is given by:

$$m_0^2 = (6-p)(7-p)\,, \tag{2.74}$$

eq. (2.72) can be written as:

$$\left(\mathcal{O}_p + m_0^2\right)\zeta_{0,m} = \text{constant}\,, \tag{2.75}$$

which means that $\zeta_{0,m}$ is a massive mode with mass m_0. Notice that, as $p < 6$, m_0^2 is strictly positive.

For a general value of $l > 0$ the equations of motion can be conveniently expressed in terms of the variables $\eta_{l,m}$ and $\zeta_{l,m}$. Indeed, by differentiating with respect to the time the first equation (2.69), and using the definition (2.67), we can put them in terms of $\eta_{l,m}$ and $\zeta_{l,m}$. Actually, if we define the mass matrix $\mathcal{M}_p$ as:

$$\mathcal{M}_p = \begin{pmatrix} l\,(l+6-p) + (7-p)\,(6-p) & p-7 \\ \\ (p-7)\,l\,(l+6-p) & l\,(l+6-p) \end{pmatrix}\,, \tag{2.76}$$

the equations of motion can be written as:

$$\Big(\mathcal{O}_p + \mathcal{M}_p\Big)\begin{pmatrix} \zeta_{l,m} \\ \eta_{l,m} \end{pmatrix} = 0\,, \tag{2.77}$$

where $\mathcal{O}_p$ is the wave operator defined in eq. (2.73). In order to check that our wrapped configurations are stable, we must verify that the eigenvalues of the matrix $\mathcal{M}_p$ are non-negative. After a simple calculation one can show that these eigenvalues are:

$$m_l^2 = \begin{cases} (l+6-p)\,(l+7-p) & \text{for } l = 0, 1, \ldots\,, \\ \\ l\,(l-1) & \text{for } l = 1, 2, \ldots\,, \end{cases} \tag{2.78}$$

where we have already included the $l = 0$ case. Eq. (2.78) proves that there are not negative mass modes in the spectrum of small oscillations for $p < 6$, which demonstrates that, as claimed, our static solutions are stable.

3. Flux stabilization of M5-branes

In this section we are going to describe a mechanism of flux stabilization in M-theory. We shall consider a particular solution of the equations of motion of eleven dimensional supergravity which is the one associated to a stack of N parallel M5-branes. The metric of this solution takes the form [21]:

$$ds^2 = \frac{r}{R}\,(-dt^2 + dx^1 + \cdots + dx_5^2) + \frac{R^2}{r^2}\,(dr^2 + r^2\,d\Omega_4^2)\,, \tag{3.1}$$

where the "radius" R is given by:

$$R^3 = \pi\,N\,l_p^3\,. \tag{3.2}$$

In eq. (3.2) l_p is the Planck length in eleven dimensions. The M5-brane solution of D=11 supergravity has also a non-vanishing value of the four-form field strength $F^{(4)}$, under which the M5-branes are magnetically charged. This field strength is given by:

$$F^{(4)} = 3\,R^3\,\epsilon_{(4)}\,. \tag{3.3}$$

It is not difficult to find a three-form potential $C^{(3)}$ such that $F^{(4)} = dC^{(3)}$. Actually, if we decompose the S^4 line element $d\Omega_4^2$ as in eq. (2.4) and use the same orientation conventions as in section 2, one can readily check that $C^{(3)}$ can be taken as:

$$C^{(3)} = -R^3\,C_4(\theta)\epsilon_{(3)}\,, \tag{3.4}$$

where $C_4(\theta)$ is the function defined in eqs. (2.5) and (2.6), namely $C_4(\theta) = \cos\theta\sin^2\theta + +2(\cos\theta - 1)$.

We will put in this background a probe M5-brane, whose action will be given by the so-called PST formalism [16]. The fields of this formalism include a three-form field strength F, whose potential is a two-form field A (i.e. $F = dA$) and a scalar field a (the PST scalar). The field strength F can be combined with (the pullback of) the background potential $C^{(3)}$ to form the field[5] H as:

$$H = F - C^{(3)}\,. \tag{3.5}$$

Let us now define the field $\tilde{H}$ as follows:

$$\tilde{H}^{ij} = \frac{1}{3!\,\sqrt{-\det g}}\,\frac{1}{\sqrt{-(\partial a)^2}}\,\epsilon^{ijklmn}\,\partial_k\,a\,H_{lmn}\,, \tag{3.6}$$

where g is the induced metric on the M5-brane worldvolume. The PST action of the M5-brane probe is:

$$\begin{aligned} S = T_{M5}\int d^6\sigma\Big[-\sqrt{-\det(g+\tilde{H})}+\frac{\sqrt{-\det g}}{4\partial a\cdot\partial a}\,\partial_i a\,({}^*H)^{ijk}\,H_{jkl}\partial^l a\Big]+\\ +T_{M5}\int\Big[C^{(6)}\,+\,\frac{1}{2}\,F\wedge\,C^{(3)}\Big]\,, \end{aligned} \tag{3.7}$$

where ${}^*H$ denotes the Hodge dual of H, $C^{(6)}$ is (the pullback of) the 6-form potential dual to $C^{(3)}$, and the M5-brane tension is given by:

$$T_{M5} = \frac{1}{(2\pi)^5\,l_p^6}\,. \tag{3.8}$$

We will extend our M5-brane probe along the directions transverse to the M5-branes of the background and along one of the directions parallel to them. Without loss of generality we will take the latter to be the x^5 direction. Accordingly, our worldvolume coordinates σ^α will be taken to be:

$$\sigma^\alpha = (t, r, x^5, \theta^1, \theta^2, \theta^3)\,, \tag{3.9}$$

and the embedding of the M5-brane probe is determined by a function $\theta = \theta(\sigma^\alpha)$. As in the case of the RR background, we shall mainly look for solutions with $\theta =$ constant, which represent a M5-brane wrapped on a three-sphere and extended along the r and x^5 directions.

As discussed in ref. [16], the scalar a is an auxiliary field which can be eliminated from the action by fixing its gauge symmetry. The price one must pay for this gauge fixing is the loss of manifest covariance. A particularly convenient choice for a is:

$$a = x^5\,. \tag{3.10}$$

[5]We hope that this field H will not be confused with the hamiltonian.

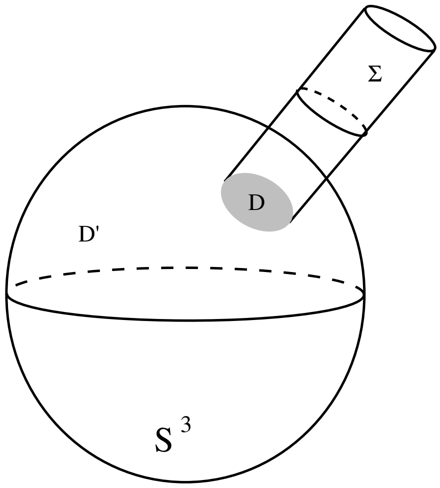

Figure 3: An M2-brane with worldvolume Σ having its boundary on the worldvolume of an M5-brane. If Σ is attached to a submanifold of the M5-brane worldvolume with the topology of S^3, there are two possible disks D and D' on the S^3 whose boundary is $\partial\Sigma$.

In this gauge the components of the worldvolume potential A with x^5 as one of its indices can be gauge fixed to zero [16]. Moreover, if we consider configurations of A and of the embedding angle θ which are independent of x^5, one readily realizes that the components of the three-forms F and H along x^5 also vanish and, as a consequence, only the square root term of the PST action (3.7) is non-vanishing. As we will verify soon this constitutes a great simplification.

3.1 Quantization condition and M5-brane configurations

In order to find stable S^3-wrapped configurations of the M5-brane probe, we need to switch on a non-vanishing worldvolume field which could prevent the collapse to one of the poles of the S^3. As in section 2 (and ref. [1]) the value of this worldvolume field is determined by some quantization condition which can be obtained by coupling the M5-brane to an open M2-brane.

Let us consider an open M2-brane with worldvolume given by a three-manifold Σ whose boundary $\partial\Sigma$ lies on the worldvolume of an M5-brane. For simplicity we shall consider the case in which $\partial\Sigma$ has only one component (see figure 3). Clearly, $\partial\Sigma$ is also the boundary of some disk D on the worldvolume of the M5-brane. Let $\hat\Sigma$ be a four-manifold whose boundaries are Σ and D, i.e. $\partial\hat\Sigma = \Sigma + D$. The coupling of the brane to the supergravity background and to the M5-brane is described by an action of the form:

$$S_{int}\,[\,\hat\Sigma, D\,] = T_{M2}\int_{\hat\Sigma} F^{(4)} + T_{M2}\int_D H\,, \tag{3.11}$$

where T_{M2} is the tension of the M2-brane, given by:

$$T_{M2} = \frac{1}{(2\pi)^2\, l_p^3}\,. \tag{3.12}$$

In a topologically trivial situation, if we represent $F^{(4)}$ as $dC^{(3)}$ and $F = dA$, the above action reduces to the more familiar expression:

$$S_{int} = T_{M2}\int_{\Sigma} C^{(3)} + T_{M2}\int_{\partial D} A\,. \tag{3.13}$$

We shall regard eq. (3.11) as the definition of the interaction term of the M2-brane action. Notice that, in general, $\hat{\Sigma}$ and D are not uniquely defined. To illustrate this point let us consider the case in which we attach the M2-brane to a M5-brane worldvolume which has some submanifold with the topology of S^3. This is precisely the situation in which we are interested in. As illustrated in figure 3, we have two possible elections for the disk in eq. (3.11) namely, we can choose the "internal" disk D or the "external" disk D'. Changing $D \rightarrow D'$, the manifold $\hat{\Sigma}$ changes to $\hat{\Sigma}'$, with $\partial\hat{\Sigma}' = \Sigma + D'$ and, in general, S_{int} also changes. However, in the quantum-mechanical theory, the action appears in a complex exponential of the form $\exp[\,i\,S_{int}\,]$. Thus, we should require that:

$$e^{iS_{int}[\hat{\Sigma},D\,]} = e^{iS_{int}[\hat{\Sigma}',D'\,]}\,. \tag{3.14}$$

The condition (3.14) is clearly equivalent to:

$$S_{int}[\hat{\Sigma}',D'] - S_{int}[\hat{\Sigma},D] = 2\pi n\,, \tag{3.15}$$

with $n \in \mathbb{Z}$. The left-hand side of eq. (3.15) can be straightforwardly computed from eq. (3.11). Actually, if $\hat{\mathcal{B}}$ is the 4-ball bounded by $D' \cup (-D) = S^3$, one has:

$$S_{int}[\hat{\Sigma}',D'] - S_{int}[\hat{\Sigma},D] = T_{M2}\int_{\hat{\mathcal{B}}} F^{(4)} + T_{M2}\int_{\partial\hat{\mathcal{B}}} H\,. \tag{3.16}$$

Using this result in eq. (3.15), we get the condition:

$$\int_{\hat{\mathcal{B}}} F^{(4)} + \int_{\partial\hat{\mathcal{B}}} H = \frac{2\pi n}{T_{M2}}\,, \qquad n \in \mathbb{Z}\,. \tag{3.17}$$

If $F^{(4)}$ can be represented as $dC^{(3)}$ on $\hat{\mathcal{B}}$, the first integral on the left-hand side of eq. (3.17) can be written as an integral of $C^{(3)}$ over $\partial\hat{\mathcal{B}} = S^3$. Our parametrization of $C^{(3)}$ (eq. (3.4)) is certainly non-singular if we are outside of the poles of the S^4. If this is the case we get the quantization condition:

$$\int_{S^3} F = \frac{2\pi n}{T_{M2}}\,, \qquad n \in \mathbb{Z}\,. \tag{3.18}$$

Eq. (3.18), which is the M-theory analogue of eq. (1.1), is the quantization condition we were looking for. It is very simple to obtain a solution of this equation. Let us take F proportional to the volume element $\epsilon_{(3)}$ of the S^3. Taking into account that the volume of the unit three-sphere is $\Omega_3 = 2\pi^2$ (see eq. (2.18)), we can write down immediately the following solution of eq. (3.18):

$$F = \frac{n}{\pi T_{M2}}\,\epsilon_{(3)}\,. \tag{3.19}$$

We can put this solution in a more convenient form if we use the following relation between the M2-brane tension and the radius R:

$$T_{M2} = \frac{N}{4\pi R^3}\,, \tag{3.20}$$

which follows from eqs. (3.2) and (3.12). By using eq. (3.20), one can rewrite eq. (3.19) as:

$$F = 4R^3\,\frac{n}{N}\,\epsilon_{(3)}\,. \tag{3.21}$$

We can use the ansatz (3.21) and the potential $C^{(3)}$ of eq. (3.4) to compute the three-form field H of eq. (3.5). It turns out that the result for H can be written in terms of the function $\mathcal{C}_{4,n}(\theta)$ defined in eq. (2.20). One gets:

$$H = R^3\,\mathcal{C}_{4,n}(\theta)\,\epsilon_{(3)}\,. \tag{3.22}$$

Let us now assume that the angle θ characterizing the M5-brane embedding only depends on the radial coordinate r and, as before, let us denote by θ' to the derivative $d\theta/dr$. As was mentioned above, in the $a = x^5$ gauge and for this kind of embedding, only the first term of the PST action (3.7) is non vanishing and, as a consequence, all the dependence on H of this action comes through the field $\tilde{H}$ defined in eq. (3.6). Actually, the only non-vanishing component of $\tilde{H}$ is:

$$\tilde{H}_{0r} = -\frac{i}{(\sin\theta)^3}\,\sqrt{\frac{R}{r}}\,\sqrt{1+r^2\theta'^2}\,\mathcal{C}_{4,n}\,(\theta)\,. \tag{3.23}$$

After a simple calculation one can obtain the induced metric g and, using eq. (3.23), the lagrangian density of the M5-brane. The result is:

$$\mathcal{L} = -T_{M5}\,R^3\,\sqrt{1+r^2\theta'^2}\,\sqrt{(\sin\theta)^6 + (\mathcal{C}_{4,n}(\theta))^2}\,. \tag{3.24}$$

Notice the close similarity of this result and the hamiltonian density of eq. (2.23) for $p = 4$, i.e. for the D4-D4 system. Indeed, it is immediate to check that the solutions with constant θ are the same in both systems, i.e. $\theta = \bar{\theta}_{4,n}$ with $0 < n < N$, where $\bar{\theta}_{4,n}$ is given in eq. (2.40) (for $n = 0, N$ we have the singular solutions with $\theta = 0, \pi$). This result is quite natural since the D4-D4 system can be obtained from the M5-M5 one by means of a double dimensional reduction along the x^5 direction. The energy density for these solutions can be easily obtained from the lagrangian (3.24). One gets:

$$\mathcal{E}_n^{M5} = \frac{n(N-n)}{N}\,T_{M2}\,, \tag{3.25}$$

which, again, closely resembles the D4-D4 energy of eq. (2.41). In particular $\mathcal{E}_n^{M5} \rightarrow n\,T_{M2}$ as $N \rightarrow \infty$, which implies that, semiclassically, our configurations can be regarded as bound states of M2-branes. Moreover, one can check that eq. (2.57) with $p = 4$ is a BPS condition for the M5-brane system. The integration of this equation can be read from eq. (2.59) and represents a baryonic vertex in M-theory [18], n being the number of M2-branes which form the baryon at $r \to \infty$. The $\theta = \bar{\theta}_{4,n}$ solution can be obtained as the $r \to 0$ limit of the M-theory baryon, in complete analogy with the analysis at the end of section 2.2.

3.2 Fluctuations and stability

We will now perturb our static solution in order to check its stability. As in section 2.3, we must allow the angle θ to deviate from $\bar{\theta}_{4,n}$ and the worldvolume field strength F to vary from the value displayed in eq. (3.21). The best way to find out which components of F must be included in the perturbation is to choose a gauge. As F in eq. (3.21) has only components along the sphere S^3, one can represent it by means of a potential $\bar{A}_{\hat{i}\,\hat{j}}$ which also has component only on S^3 (in what follows indices along S^3 will be denoted with a hat). Accordingly, the perturbation of F will be parametrized as a fluctuation of the S^3-components of the potential A. Thus, we put:

$$\theta = \bar{\theta}_{4,n} + \xi\,, \qquad A_{\hat{i}\,\hat{j}} = \bar{A}_{\hat{i}\,\hat{j}} + \alpha_{\hat{i}\,\hat{j}}\,, \tag{3.26}$$

where ξ and $\alpha_{\hat{i}\,\hat{j}}$ are small. For simplicity we shall assume that ξ and the $\alpha_{\hat{i}\,\hat{j}}$'s do not depend on x^5. Using the parametrization of A in eq. (3.26), it is clear that the S^3-components of the three-form field H can be written as:

$$H_{\hat{i}\,\hat{j}\,\hat{k}} = R^3\,[\mathcal{C}_{4,n}(\theta) + f]\,\frac{\epsilon_{\hat{i}\,\hat{j}\,\hat{k}}}{\sqrt{\hat{g}}}\,, \tag{3.27}$$

where f can be put in terms of derivatives of the type $\partial_{\hat{i}}\,\alpha_{\hat{j}\,\hat{k}}$. In eq. (3.27) $\hat{g}$ is the determinant of the metric of the S^3 and we are using the convention $\epsilon^{\hat{1}\,\hat{2}\,\hat{3}} = \epsilon_{\hat{1}\,\hat{2}\,\hat{3}}/\hat{g} = 1$. As $\alpha_{\hat{i}\,\hat{j}}$ in (3.26) depends on t and r, it follows that we have now non-zero components $H_{0\hat{i}\,\hat{j}} = \partial_0\,\alpha_{\hat{i}\,\hat{j}}$ and $H_{r\hat{i}\,\hat{j}} = \partial_r\,\alpha_{\hat{i}\,\hat{j}}$. Thus, in the gauge (3.10), the non-vanishing components of $\tilde{H}$ are $\tilde{H}_{0r}$, $\tilde{H}_{0\hat{i}}$ and $\tilde{H}_{r\hat{i}}$. To the relevant order, these components take the values:

$$\begin{aligned}
\tilde{H}_{0r} &= -i\sqrt{\frac{R}{r}}\cot\bar{\theta} + \frac{i}{(\sin\bar{\theta})^2}\sqrt{\frac{R}{r}}\Big(3\xi - \frac{f}{\sin\bar{\theta}}\Big) - 3i\frac{\cos\bar{\theta}}{(\sin\bar{\theta})^3}\sqrt{\frac{R}{r}}\Big(2\xi^2 - \xi\frac{f}{\sin\bar{\theta}}\Big) + \\
&\quad + \frac{i}{2}R^2\cot\bar{\theta}\left[\sqrt{\frac{R^3}{r^3}}\,(\partial_t\xi)^2 - \sqrt{\frac{r^3}{R^3}}\,(\partial_r\xi)^2\right] + \frac{i}{2}\sqrt{\frac{R}{r}}\frac{\cos\bar{\theta}}{(\sin\bar{\theta})^3}\,(\partial_{\hat{i}}\xi)^2\,, \\
\tilde{H}_{0\hat{i}} &= \frac{i}{2R\sin\bar{\theta}}\sqrt{\frac{r^3}{R^3}}\,\hat{g}_{\hat{i}\,\hat{j}}\,\frac{\epsilon^{\hat{j}\,\hat{l}\,\hat{m}}}{\sqrt{\hat{g}}}\,H_{r\,\hat{l}\,\hat{m}}\,, \\
\tilde{H}_{r\hat{i}} &= \frac{i}{2R\sin\bar{\theta}}\sqrt{\frac{R^3}{r^3}}\hat{g}_{\hat{i}\,\hat{j}}\,\frac{\epsilon^{\hat{j}\,\hat{l}\,\hat{m}}}{\sqrt{\hat{g}}}\,H_{0\,\hat{l}\,\hat{m}}\,,
\end{aligned} \tag{3.28}$$

with $\bar\theta \equiv \bar\theta_{4,n}$. Using these results we can compute the lagrangian for the fluctuations. After some calculation one arrives at:

$$\begin{aligned}\mathcal{L} = & -R^3\, T_{M5}\, \cos\bar\theta\, f + R^3\, T_{M5}\, (\sin\bar\theta\,)^2 \times \\ & \times\frac{1}{2}\Bigg[R^3 r^{-1}(\partial_t\xi)^2 - r^2\,(\partial_r\xi)^2 - (\partial_{\hat i}\xi)^2 + \frac{1}{2R^3 r(\sin\bar\theta)^2}\,(H_{0\hat j\,\hat k})^2 - \\ & \qquad - \frac{r^2}{2R^6(\sin\bar\theta)^2}\,(H_{r\hat j\,\hat k})^2 - 6\xi^2 - \frac{f^2}{(\sin\bar\theta)^2} + 6\,\frac{f\xi}{\sin\bar\theta}\Bigg]\,,\end{aligned} \tag{3.29}$$

where $(H_{0\hat j\,\hat k})^2$ and $(H_{r\hat j\,\hat k})^2$ are contractions with the metric of the S^3. In eq. (3.29) we have kept terms up to second order and we have dropped the zero-order term.

The analysis of the equations of motion derived from eq. (3.29) is similar to the one performed in section 2.3. For this reason we will skip the details and will give directly the final result. Let us expand f and ξ is spherical harmonics of S^3 as in eq. (2.65) and let $f_{l,m}(t,r)$ and $\zeta_{l,m}(t,r)$ denote their modes respectively. The equations of motion of these modes can be written as:

$$\Big(R^3 r^{-1}\partial_0^2 - \partial_r\, r^2\,\partial_r\, + \mathcal{M}_4\Big)\begin{pmatrix}\zeta_{l,m}\\ \frac{f_{l,m}}{\sin\bar\theta}\end{pmatrix} = 0\,, \tag{3.30}$$

where the mass matrix $\mathcal{M}_4$ is the same as that corresponding the D4-D4 system (i.e. the one of eq. (2.76) for $p = 4$). Notice that the wave operator on the left-hand side of eq. (3.30) is formally the same as $\mathcal{O}_4$ in eq. (2.73) (although the radius R is not the same quantity in both cases). Thus, the eigenvalues of the mass matrix are non-negative and, actually, the same as in the D4-D4 system. Therefore our static M-theory configurations are indeed stable.

4. The D3-brane in a (p, q) fivebrane background

We are now going to study the motion of a D3-brane probe in a background of a stack of fivebranes which are charged under both the NS and RR three-form fields strengths of type-IIB supergravity. This background was obtained in ref. [19] by exploiting the S-duality of type-IIB supergravity and is characterized by two coprime integers p and q, and we will refer to it as the (p, q) fivebrane background. It can be regarded as the one created by an object which is a bound state of p NS5-branes and q D5-branes. In particular, for $(p, q) = (1, 0)$ the corresponding NS5-D3 system is the analogue of the one studied in ref. [1] in the type-IIB theory. If, on the other hand, we take $(p, q) = (0, 1)$ we recover the D5-D3 problem studied in section 2.

In order to describe the background, following ref. [19], let us introduce some notations. First of all we define the quantity:

$$\mu_{(p,q)} = p^2 + (q - p\chi_0)^2\, g_s^2\,, \tag{4.1}$$

where χ_0 is the asymptotic value of the RR scalar. The "radius" $R_{(p,q)}$ for a stack of N (p,q) fivebranes is defined in terms of $\mu_{(p,q)}$ as:

$$R^2_{(p,q)} = N\left[\mu_{(p,q)}\right]^{1/2}\alpha' \,. \tag{4.2}$$

We will use $R_{(p,q)}$ to define the near-horizon harmonic function:

$$H_{(p,q)}(r) = \frac{R^2_{(p,q)}}{r^2} \,. \tag{4.3}$$

The near-horizon metric, in the string frame, for the stack of (p,q) fivebranes can be written as:

$$ds^2 = \left[h_{(p,q)}(r)\right]^{-1/2}\left[\left[H_{(p,q)}(r)\right]^{-1/4}(-dt^2 + dx^2_{\|}) + \left[H_{(p,q)}(r)\right]^{3/4}(dr^2 + r^2\, d\Omega_3^2)\right], \tag{4.4}$$

where the function $h_{(p,q)}(r)$ is given by:

$$h_{(p,q)}(r) = \frac{\mu_{(p,q)}}{p^2\left[H_{(p,q)}(r)\right]^{1/2} + (q - p\chi_0)^2\, g_s^2\left[H_{(p,q)}(r)\right]^{-1/2}} \,. \tag{4.5}$$

To simplify the equations that follow we shall take from now on $g_s = 1$ and $\chi_0 = 0$. (The dependence on g_s and χ_0 can be easily restored). Other fields of this background include the dilaton:

$$e^{-\phi} = h_{(p,q)}(r) \,, \tag{4.6}$$

and the RR scalar:

$$\chi = \frac{pq}{\mu_{(p,q)}}\left(\left[H_{(p,q)}(r)\right]^{1/2} - \left[H_{(p,q)}(r)\right]^{-1/2}\right) h_{(p,q)}(r) \,. \tag{4.7}$$

In addition we have non-zero NS and RR three-form field strengths. Let us call B and $C^{(2)}$ to their two-form potentials respectively. If we take coordinates on the three-sphere as in eq. (2.4), this potentials can be taken as:

$$B = -pN\alpha'\, C_5(\theta)\, \epsilon_{(2)} \,, \qquad C^{(2)} = -qN\alpha'\, C_5(\theta)\, \epsilon_{(2)} \,, \tag{4.8}$$

where $C_5(\theta)$ is the function defined in eqs. (2.5) and (2.6), i.e. $C_5(\theta) = \sin\theta\cos\theta - \theta$.

The action of a D3-brane probe in the above background is the sum of the Dirac-Born-Infeld and Wess-Zumino terms. The latter now includes the coupling of the brane to the RR potential $C^{(2)}$ and to the RR scalar χ:

$$S = -T_3\int d^4\sigma\, e^{-\phi}\sqrt{-\det(g + \mathcal{F})} + T_3\int\left[\mathcal{F}\wedge C^{(2)} + \frac{1}{2}\chi\mathcal{F}\wedge\mathcal{F}\right], \tag{4.9}$$

with $\mathcal{F}$ being:

$$\mathcal{F} = dA - B = F - B \,. \tag{4.10}$$

4.1 Quantization conditions

The analysis of the action (4.9) for the (p,q) fivebrane background was performed in ref. [23] (for the NS5-D3 system see refs. [24, 25]). Here we shall choose our worldvolume coordinates as in (2.11) and we will look for solutions of the equations of motion with constant θ. Notice that, as now our background contains non-zero NS and RR forms, it is natural to expect that the worldvolume gauge field F has both electric and magnetic components. The latter can be determined by means of the flux quantization condition (1.1), whereas the electric wordlvolume field is constrained by the condition (2.16). Accordingly we shall first require that:

$$\int_{S^2} F = \frac{2\pi n_1}{T_f}\,, \tag{4.11}$$

with $n_1 \in \mathbb{Z}$. It is rather simple to solve this condition. We only have to take F as:

$$F = \pi n_1 \alpha' \epsilon_{(2)} + F_{0,r} dt \wedge dr\,, \tag{4.12}$$

where we have assumed that the electric worldvolume field has only components along the radial direction. By using the definition of $\mathcal{F}$ in eq. (4.10) and the expression of the B field in eq. (4.8), one easily verifies that eq. (4.12) is equivalent to the following expression for $\mathcal{F}$:

$$\mathcal{F} = f_{12}(\theta)\epsilon_{(2)} + F_{0,r} dt \wedge dr\,, \tag{4.13}$$

with $f_{12}(\theta)$ being:

$$f_{12}(\theta) \equiv pN\alpha' C_5(\theta) + \pi n_1 \alpha'\,. \tag{4.14}$$

As in our previous examples, let us assume that the angle θ depends only on the radial coordinate r. By substituting the ansatz (4.13) in eq. (4.9), one can find the form of the lagrangian density:

$$\begin{aligned}\mathcal{L}(\theta, F) = -T_3 \Bigg[& \sqrt{r^4 \Big[H_{(p,q)}(r)\Big]^{\frac{3}{2}} (\sin\theta)^4 + e^{-\phi} f_{12}(\theta)^2}\ \times \\ & \times \sqrt{\Big[H_{(p,q)}(r)\Big]^{\frac{1}{2}} (1 + r^2\theta'^2) - e^{-\phi} F_{0,r}^2} + \big(qN\alpha' C_5(\theta) - \chi f_{12}(\theta)\big) F_{0,r} \Bigg]\,.\end{aligned} \tag{4.15}$$

Next, we make use of the electric quantization condition (2.16) and require that:

$$\int_{S^2} d^2\theta \sqrt{\hat g} \frac{\partial \mathcal{L}}{\partial F_{0,r}} = n_2 T_f\,, \tag{4.16}$$

where n_2 is another integer. By plugging the lagrangian density (4.15) into (4.16), one can obtain $F_{0,r}$ as a function of $\theta(r)$ and of the integers n_1 and n_2. Actually, one can eliminate in this way $F_{0,r}$ from the expression of the hamiltonian H, which can be obtained from $\mathcal{L}$ by means of a Legendre transformation (see eq. (2.22)). The resulting hamiltonian can be put in the form:

$$H = T_3\Omega_2 \int dr\sqrt{1+r^2\theta'^2} \times \tag{4.17}$$
$$\times\sqrt{R^4_{(p,q)}(\sin\theta)^4 + [\mu_{(p,q)}]^{-1}[(pf_{12}(\theta)+q\Pi(\theta))^2 + H_{(p,q)}(r)(qf_{12}(\theta)-p\Pi(\theta))^2]}\,,$$

where $\Pi(\theta)$ is the function:

$$\Pi(\theta) \equiv qN\alpha' C_5(\theta) + \pi n_2\alpha'\,. \tag{4.18}$$

The solutions of the equations of motion with θ = constant can be obtained by solving the equation $\partial H/\partial\theta = 0$ for $\theta' = 0$. A glance at the right-hand side of eq. (4.17) reveals immediately that these solutions only exist if the r-dependent term inside the square root in (4.17) is zero. We thus get the condition:

$$qf_{12}(\theta) = p\Pi(\theta)\,, \tag{4.19}$$

which, after using eqs. (4.14) and (4.18), is equivalent to the following relation between the integers n_1 and n_2:

$$qn_1 = pn_2\,. \tag{4.20}$$

But, as p and q are coprime integers, the only possibility to fulfill eq. (4.20) is that n_1 and n_2 be of the form:

$$n_1 = p\,n\,, \qquad n_2 = qn\,, \tag{4.21}$$

with $n \in \mathbb{Z}$. Thus our two quantization integers n_1 and n_2 are not independent and they can be put in terms of another integer n. By using the relations (4.21), one can rewrite $f_{12}(\theta)$ and $\Pi(\theta)$ in terms of n:

$$f_{12}(\theta) = pN\alpha'\,\mathcal{C}_{5,n}(\theta)\,, \qquad \Pi(\theta) = qN\alpha'\,\mathcal{C}_{5,n}(\theta)\,, \tag{4.22}$$

where $\mathcal{C}_{5,n}(\theta)$ is the function defined in eq. (2.20). If we now substitute these expressions into the hamiltonian (4.17), we get the following expression of H:

$$H = T_3\,\Omega_2\,R^2_{(p,q)}\int dr\,\sqrt{1+r^2\theta'^2}\,\sqrt{(\sin\theta)^4 + \Big(\mathcal{C}_{5,n}(\theta)\Big)^2}\,. \tag{4.23}$$

Apart from a global coefficient, this hamiltonian is the same as the one in eq. (2.23) for the D5-brane background. Thus, the energy is clearly minimized for $\theta = \bar\theta_{5,n}$, where the $\bar\theta_{5,n}$'s are the angles written in eq. (2.37) . The energy densities for these angles are easily computed from eq. (4.23). One gets:

$$\mathcal{E}^{(p,q)}_{5,n} = \frac{NT_{(q,p)}}{\pi}\sin\Big[\frac{n}{N}\,\pi\Big]\,, \tag{4.24}$$

where $T_{(q,p)}$ is the tension of the (q,p)-string which, for arbitrary values of g_s and χ_0, is given by:

$$T_{(q,p)} = \sqrt{(q - p\chi_0)^2 + \frac{p^2}{g_s^2}}\,T_f\,. \tag{4.25}$$

By comparing eqs. (2.38) and (4.24) it follows that $\mathcal{E}_{5,n}^{(p,q)}$ can be obtained from $\mathcal{E}_{5,n}$ by substituting T_f by $T_{(q,p)}$. In particular, if $N \to \infty$ the energy density $\mathcal{E}_{5,n}^{(p,q)}$ equals $nT_{(q,p)}$ and, thus, the configurations we have found can be regarded as bound states of n (q,p)-strings. It is also easy to get the worldvolume electric field $\bar{F}_{0,r}$ of our solutions. It takes the form:

$$\bar{F}_{0,r} = \frac{(q - p\chi_0)g_s}{\sqrt{p^2 + (q - p\chi_0)^2 g_s^2}}\cos\left[\frac{n}{N}\,\pi\right]. \tag{4.26}$$

It is also clear from the expression of the hamiltonian in (4.23) that one can represent it as in eq. (2.53) and, as a consequence, one can find a bound for the energy whose saturation gives rise to a BPS condition. As the only difference between the hamiltonian (4.23) and that of eq. (2.23) for the D5-brane is a global coefficient, it follows that the BPS differential equation is just the one displayed in eq. (2.57) for $p = 5$. Its solution is given in eq. (2.59) and, again, includes our wrapped configurations as particular cases. Moreover, these $\theta = \bar{\theta}_{p,n}$ configurations can be regarded as the $r \to 0$ limit of the general solution. Actually, in ref. [23] the condition (4.19) and the form (4.23) of the hamiltonian were obtained by using the S-duality of the worldvolume action. It was also checked in this reference that the D3-brane configurations which saturate the bound preserve 1/4 of the bulk supersymmetry.

4.2 Stability

The static configurations of the D3-brane studied above are stable under small perturbations, as one can check following the same steps as in sections 2.3 and 3.2. First of all, we parametrize the angle fluctuations as:

$$\theta = \bar{\theta}_{5,n} + \xi\,, \tag{4.27}$$

whereas the gauge field fluctuates as:

$$\mathcal{F} = [\,f_{12}(\theta) + g\,]\,\epsilon_{(2)} + [\bar{F}_{0,r} + f\,]\,dt \wedge dr\,. \tag{4.28}$$

The angle fluctuation ξ and the electric (magnetic) field fluctuation f (g) are supposed to be small and only terms up to second order are retained in the lagrangian. The corresponding equations of motion involve now the wave operator $\mathcal{O}_5^{(p,q)}$, which acts on any function ψ as:

$$\mathcal{O}_5^{(p,q)}\,\psi \;\equiv\; R^2_{(p,q)}\,\partial_0^2\,\psi - \partial_r(r^2\partial_r\psi)\,. \tag{4.29}$$

Notice that $\mathcal{O}_5^{(p,q)}$ is obtained from $\mathcal{O}_5$ in eq. (2.73) by means of the substitution

$R \to R_{(p,q)}$. Let us combine ξ, f and g into the field η, defined as:

$$\eta \equiv \frac{1}{p^2 H_{(p,q)}(r) + q^2 \sin^2 \bar{\theta}_{5,n}} \Big[[\mu_{(p,q)}]^{1/2} \Big(q \sin \bar{\theta}_{5,n} f - \frac{p}{r^2} g \Big) + 2q^2 \sin^2 \bar{\theta}_{5,n} \xi \Big] \,, \quad (4.30)$$

and let us expand ξ and η is spherical harmonics of S^2. If $\zeta_{l,m}$ and $\eta_{l,m}$ denote their modes respectively, one can prove after some calculation that the equations of motion for $\zeta_{l,m}$ and $\eta_{l,m}$ can be written as:

$$\Big(\mathcal{O}_5^{(p,q)} + \mathcal{M}_5 \Big) \begin{pmatrix} \zeta_{l,m} \\ \eta_{l,m} \end{pmatrix} = 0 \,. \quad (4.31)$$

In eq. (4.31) $\mathcal{M}_5$ is the matrix defined in eq. (2.76), whose eigenvalues, as proved in section 2.3, are always non-negative. There is also a decoupled mode σ, whose expression in terms of ξ, f and g is:

$$\sigma \equiv \frac{\sin \bar{\theta}_{5,n}}{r[p^2 H_{(p,q)}(r) + q^2 \sin^2 \bar{\theta}_{5,n}]} \Big[[\mu_{(p,q)}]^{1/2} (q \sin \bar{\theta}_{5,n} g - p\, R^2_{(p,q)}\, f\,) - 2pq\, R^2_{(p,q)} \sin \bar{\theta}_{5,n} \xi \Big] \,. \quad (4.32)$$

The equation of motion of σ can be written as:

$$\Big(\mathcal{O}_5^{(p,q)} + l(l+1) \Big)\, \sigma_{l,m} = 0 \,, \quad (4.33)$$

where $\sigma_{l,m}$ are the modes of the expansion of σ in S^2-spherical harmonics. It is evident from eq. (4.33) that the mass eigenvalues of $\sigma_{l,m}$ are non-negative, which confirms that the configurations around which we are expanding are stable.

5. Summary and discussion

In this paper we have studied certain configurations of branes which are partially wrapped on spheres. These spheres are placed on the transverse region of some supergravity background, and their positions, characterized by a polar angle which measures their latitude in a system of spherical coordinates, are quantized and given by a very specific set of values. We have checked that our configurations are stable by analyzing their behaviour under small fluctuations and, by studying their energy, we concluded that they can be regarded as a bound state of strings or, in the case of the M5-M5 system, M2-branes. We have verified this fact explicitly in appendix B for the case of a wrapped D3-brane in the background of a NS5-brane. Indeed, we have proved that, by embedding a D1-brane in a fuzzy two-sphere in the NS5-brane background, one obtains exactly the same energies and allowed polar angles as for a wrapped D3-brane in the same geometry. Clearly, a similar description of all the cases studied here would be desirable and would help to understand more precisely the rôle of noncommutative geometry in the formation of these bound states. In this sense it is interesting to point out that the polarization of multiple fundamental strings in a RR background was studied in ref. [26]

Contrary to ref. [1], the problems treated here do not have a CFT description to compare with. Thus, we do not know to what extent we can trust our Born-Infeld results. However, one could argue that we have followed the same methodology as in ref. [1] and, actually, our configurations can be connected to the ones in [1] by string dualities. Moreover, the BPS nature of our configurations make us reasonably confident of the correctness of our conclusions.

The presence of a non-trivial supergravity background is of crucial importance in our analysis. Indeed, these backgrounds induce worldvolume gauge fields on the brane probes, which prevent their collapse. The stabilization mechanisms found here are a generalization of the one described in refs. [1, 2], and are based on a series of quantization rules which determine the values of the worldvolume gauge fields. We have reasons to believe that our results are generic and can be extended to other geometries such as, for example, the ones generated by the Dp-D$(p-2)$ bound states [27]. Another interesting question is the implications of our results in a holographic description of gauge theories. In our opinion the study of these topics could enrich our knowledge of the brane interaction dynamics.

Acknowledgments

We are grateful to J.L. Barbon, C. Gomez, J. Mas, T. Ortin and J. M. Sanchez de Santos for discussions. This work was supported in part by DGICYT under grant PB96-0960, by CICYT under grant AEN99-0589-CO2 and by Xunta de Galicia under grant PGIDT00-PXI-20609.

A. The functions $\Lambda_{p,n}(\theta)$

In this appendix we collect the expressions of the functions $\Lambda_{p,n}(\theta)$ for $0 \le p \le 5$. They are:

$$\begin{aligned}
\Lambda_{0,n}(\theta) &= -\frac{2}{5}\left[\cos\theta\Big(3\sin^4\theta + 4\sin^2\theta + 8\Big) + 8\Big(2\frac{n}{N} - 1\Big)\right],\\
\Lambda_{1,n}(\theta) &= -\frac{5}{4}\left[\cos\theta\Big(\sin^3\theta + \frac{3}{2}\sin\theta\Big) + \frac{3}{2}\Big(\frac{n}{N}\pi - \theta\Big)\right],\\
\Lambda_{2,n}(\theta) &= -\frac{4}{3}\left[\cos\theta\Big(\sin^2\theta + 2\Big) + 2\Big(2\frac{n}{N} - 1\Big)\right],\\
\Lambda_{3,n}(\theta) &= -\frac{3}{2}\left[\cos\theta\sin\theta + \frac{n}{N}\pi - \theta\right],\\
\Lambda_{4,n}(\theta) &= -2\left[\cos\theta + 2\frac{n}{N} - 1\right],\\
\Lambda_{5,n}(\theta) &= \theta - \frac{n}{N}\pi\,.
\end{aligned}\tag{A.1}$$

The functions $\mathcal{C}_{p,n}(\theta)$ and $C_p(\theta)$ can be easily obtained from (A.1) by using their relation with the $\Lambda_{p,n}(\theta)$'s (see eqs. (2.25) and (2.20)).

B. D-string polarization in a NS5-brane background

In this appendix we will show how one can represent the wrapped branes studied in the main text as a bound state of strings. We will make use of the Myers polarization mechanism [20], in which the strings are embedded in a noncommutative space. Actually, we will only consider a particular case of those analyzed in sects. 2-4, namely the one of section 4 with $p = 1$, $q = \chi_0 = 0$, i.e. the D3-brane in the background of the NS5-brane. For convenience we will choose a new set of coordinates to parametrize the space transverse to the NS5. Instead of using the radial coordinate r and the three angles θ^1, θ^2 and θ (see eq. (2.4)), we will work with four cartesian coordinates z, x^1, x^2, x^3, which, in terms of the spherical coordinates, are given by:

$$
\begin{aligned}
z &= r\cos\theta\,,\\
x^1 &= r\sin\theta\cos\theta^2\,,\\
x^2 &= r\sin\theta\sin\theta^2\cos\theta^1\,,\\
x^2 &= r\sin\theta\sin\theta^2\sin\theta^1\,.
\end{aligned}
\tag{B.1}
$$

Conversely, r and θ can be put in terms of the new coordinates as follows:

$$
\begin{aligned}
r &= \sqrt{(z)^2+(x^1)^2+(x^2)^2+(x^3)^2}\\
\tan\theta &= \frac{\sqrt{(x^1)^2\,+(x^2)^2\,+(x^3)^2}}{z}\,.
\end{aligned}
\tag{B.2}
$$

In what follows some of our expressions will contain r and θ. It should be understood that they are given by the functions of (z, x^i) written in eq. (B.2). The near-horizon metric and the dilaton for a stack of N NS5-branes are (see eqs. (4.4) and (4.6)):

$$
\begin{aligned}
ds^2 &= -dt^2 + dx_{\|}^2 + \frac{N\alpha'}{r^2}\Big((dz)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\Big)\,,\\
e^{-\phi} &= \frac{r}{\sqrt{N\alpha'}}\,.
\end{aligned}
\tag{B.3}
$$

Moreover, the non-vanishing components of the B field in the new coordinates can be obtained from eq. (4.8). They are:

$$
B_{x^ix^j} = N\alpha'\,\frac{C_5(\theta)}{r^3\sin^3\theta}\,\epsilon_{ijk}\,x^k\,.
\tag{B.4}
$$

According to our analysis of section 4, the wrapped D3-brane in this background can be described as a bound state of D1-branes. Thus it is clear that we must consider a system of n D1-branes, moving in the space transverse to the stack of N Neveu-Schwarz fivebranes. We will employ a static gauge where the two worldsheet coordinates will be identified with t and z. The Myers proposal for the action of this system is:

$$
S_{D1} = -T_1\int dt dz\,\mathrm{STr}\Big[e^{-\phi}\sqrt{-\det\Big(P[E_{ab}+E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}]+\lambda F_{ab}\Big)\det\Big(Q^i_j\Big)}\Big]\,,
\tag{B.5}
$$

where we are adopting the conventions of ref. [20]. In eq. (B.5) $\lambda = 2\pi\alpha' = 1/T_f$, F_{ab} is the worldsheet gauge field strength (which we will assume that is zero in our case), P denotes the pullback of the spacetime tensors to the D1-brane worldsheet and STr represents the Tseytlin symmetrized trace of matrices [28]. The indices $a, b, \ldots$ correspond to directions parallel to the worldsheet (i.e. to t and z), whereas $i, j \ldots$ refer to directions transverse to the D1-brane probe. The tensor $E_{\mu\nu}$ is defined as:

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \,, \tag{B.6}$$

where $G_{\mu\nu}$ is the background metric. Let ϕ^i denote the transverse scalar fields, which are matrices taking values in the adjoint representation of $\mathrm{U}(n)$. Then $Q^i_{\;j}$ is defined as:

$$Q^i_{\;j} = \delta^i_{\;j} + i\lambda[\phi^i \,,\, \phi^k] E_{kj} \,. \tag{B.7}$$

As in ref. [20], transverse indices are raised with E^{ij}, where E^{ij} denotes the inverse of E_{ij}, i.e. $E^{ik}E_{kj} = \delta^i_{\;j}$.

Let us now make the standard identification between the transverse coordinates x^i and the scalar fields ϕ^i, namely:

$$x^i = \lambda\, \phi^i \,. \tag{B.8}$$

Notice that, after the identification (B.8), the x^i's become noncommutative coordinates represented by matrices. Actually, as in ref. [20], we will make the following ansatz for the scalar fields:

$$\phi^i = \frac{f}{2}\, \alpha^i \,, \tag{B.9}$$

where f is a c-number to be determined and the α^i's are $n\times n$ matrices corresponding to the n-dimensional irreducible representation of su(2):

$$[\alpha^i \,,\, \alpha^j] = 2i\epsilon_{ijk}\, \alpha^k \,. \tag{B.10}$$

As the quadratic Casimir of the n-dimensional irreducible representation of su(2) is $n^2 - 1$, we can write:

$$(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 = (n^2 - 1)\, I_n \,, \tag{B.11}$$

where I_n is the $n \times n$ unit matrix. By using eqs. (B.8) and (B.9) in (B.11), we get:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = \frac{\lambda^2 f^2}{4}\, (n^2 - 1)\, I_n \,, \tag{B.12}$$

which shows that, with our ansatz, the x^i's are coordinates of a fuzzy two-sphere of radius $\lambda\, f\, \sqrt{n^2-1}/2$. On the other hand, if we treat the x^i's as commutative coordinates, it is easy to conclude from eqs. (B.1) and (B.2) that the left-hand side of (B.12) is just $(r\sin\theta)^2$. In view of this, when the x^i's are non-commutative we should identify the expression written in eq. (B.12) with $(r\sin\theta)^2\, I_n$. Thus, we put:

$$\frac{f}{2} = \frac{r\sin\theta}{\lambda\sqrt{n^2-1}}\,. \tag{B.13}$$

Notice that, as can be immediately inferred from eq. (B.2), r and θ depend on the x^i's through the sum $\sum_i (x^i)^2$, which is proportional to the su(2) quadratic Casimir. Then, as matrices, r and θ are multiple of the unit matrix and, thus, we can consider them as commutative coordinates. This, in particular, means that the elements of the metric tensor $G_{\mu\nu}$ are also commutative, whereas, on the contrary, the components of the B field have a non-trivial matrix structure. By substituting our ansatz in eqs. (B.3) and (B.4), we get the following expression for the transverse components of the $E_{\mu\nu}$ tensor:

$$E_{ij} = \frac{N\alpha'}{r^2}\left[\delta^i_{\ j} + \frac{1}{\sqrt{n^2-1}}\frac{C_5(\theta)}{\sin^2\theta}\epsilon_{ijk}\,\alpha^k\right]. \tag{B.14}$$

The quantities $Q^i_{\ j}$, defined in eq. (B.7), can be readily obtained from eq. (B.14), namely:

$$Q^i_{\ j} = \left(1+\frac{N}{\pi}\frac{C_5(\theta)}{\sqrt{n^2-1}}\right)\delta^i_{\ j} - \frac{N}{\pi}\frac{C_5(\theta)}{(n^2-1)^{3/2}}\,\alpha^j\,\alpha^i - \frac{N}{\pi}\frac{\sin^2\theta}{n^2-1}\epsilon_{ijk}\,\alpha^k\,. \tag{B.15}$$

In order to compute the pullback appearing in the first determinant of the right-hand side of eq. (B.5), we need to characterize the precise embedding of the D1-brane in the transverse non-commutative space. Actually, it is straightforward to write our ansatz for the x^i's as:

$$x^i = z\,\frac{\tan\theta}{\sqrt{n^2-1}}\alpha^i\,. \tag{B.16}$$

Moreover, the kind of configurations we are looking for have constant θ angle. Thus, eq. (B.16) shows that, in this case, the x^i's are linear functions of the worldsheet coordinate z. By using this result it is immediate to find the expression of the first determinant in (B.5). One gets:

$$-\det\Big(P[E_{ab} + E_{ai}\,(Q^{-1}-\delta)^{ij}\,E_{jb}]\Big) = \frac{N\alpha'}{r^2} + \frac{\tan^2\theta}{n^2-1}\alpha^i\left[Q^{-1}\right]_{ij}\alpha^j\,, \tag{B.17}$$

where Q^{-1} satisfies $Q^{ij}\left[Q^{-1}\right]_{jk} = \delta^i_{\ k}$ with Q^{ij} being:

$$Q^{ij} = E^{ij} + i\lambda\,[\phi^i\,,\phi^j]\,. \tag{B.18}$$

As expected on general grounds, a system of D-strings can model a D3-brane only when the number n of D-strings is very large. Thus, if we want to make contact with our results of section 4, we should consider the limit in which $n\to\infty$ and keep only the leading terms in the $1/n$ expansion. Therefore, it this clear that, in this limit, we can replace n^2-1 by n^2 in all our previous expressions. Moreover, as argued in ref. [20], the leading term in a symmetrized trace of α's of the form $\mathrm{STr}((\alpha^i\alpha^i)^m)$ is $n(n^2)^m$. Then, at leading order in $1/n$, one can make the following replacement inside a symmetrized trace:

$$\alpha^i\alpha^i \;\rightarrow\; n^2\, I_n\,. \tag{B.19}$$

With this substitution the calculation of the action (B.5) drastically simplifies. So, for example, by using (B.15), one can check that, in the second term under the square root of (B.5), we should make the substitution:

$$\det\Big(Q^i_j\Big) \rightarrow \Big(\frac{N}{\pi n}\Big)^2\Big[(\sin\theta)^4 + (C_5(\theta)+\frac{\pi n}{N})^2\Big] I_n\,. \tag{B.20}$$

Moreover, as $\mathcal{C}_{5,n}(\theta) = C_5(\theta)+\pi n/N$, eq. (B.20) is equivalent to:

$$\det\Big(Q^i_j\Big) \rightarrow \Big(\frac{N}{\pi n}\Big)^2\Big[(\sin\theta)^4 + (\mathcal{C}_{5,n}(\theta))^2\Big] I_n\,. \tag{B.21}$$

We must now perform the substitution (B.19) on the right-hand side of eq. (B.17). First of all, we must invert the matrix of eq. (B.18). Actually, it is not difficult to obtain the expression of E^{ij}. After some calculation one gets:

$$E^{ij} = \frac{r^2}{N\alpha'}\,\frac{\sin^4\theta}{\sin^4\theta+(C_5(\theta))^2}\Big[\delta^i_j \;+\; \frac{(C_5(\theta))^2}{n^2\sin^4\theta}\,\alpha^i\,\alpha^j \;-\; \frac{C_5(\theta)}{n\sin^2\theta}\,\epsilon_{ijk}\,\alpha^k\Big]. \tag{B.22}$$

Plugging this result on the right-hand side of eq. (B.18), and adding the commutator of the scalar fields, one immediately obtains Q^{ij}. By inverting this last matrix one arrives at the following expression of $[Q^{-1}]_{ij}$:

$$[Q^{-1}]_{ij} = \frac{N\alpha'}{r^2}\,\frac{\sin^4\theta+(C_5(\theta))^2}{(1+a^2)\sin^4\theta}\Big[\delta^i_j \;+\; \frac{a^2-b}{n^2(1+b)}\,\alpha^i\,\alpha^j \;+\; \frac{a}{n}\,\epsilon_{ijk}\,\alpha^k\Big], \tag{B.23}$$

where, at leading order, a and b are given by:

$$a = \frac{N}{\pi n\sin^2\theta}\Big[\sin^4\theta + C_5(\theta)\,\mathcal{C}_{5,n}(\theta)\Big], \qquad b = \frac{(C_5(\theta))^2}{\sin^4\theta}\,. \tag{B.24}$$

By contracting $[Q^{-1}]_{ij}$ with $\alpha^i\alpha^j$ and applying the substitution (B.19), one gets a remarkably simple result:

$$\alpha^i\Big[Q^{-1}\Big]_{ij}\alpha^j \;\rightarrow\; n^2\,\frac{N\alpha'}{r^2}\,I_n\,. \tag{B.25}$$

By using eq. (B.25), one immediately concludes that we should make the following substitution:

$$-\det\Big(P[E_{ab}+E_{ai}\,(Q^{-1}-\delta)^{ij}\,E_{jb}]\Big) \;\rightarrow\; \frac{N\alpha'}{r^2\cos^2\theta}\,I_n\,. \tag{B.26}$$

It is now straightforward to find the action of the D1-branes in the large n limit. Indeed, by using eqs. (B.21) and (B.26), one gets:

$$S_{D1} = -T_1 \int dtdz \, \frac{N}{\pi \cos\theta} \sqrt{(\sin\theta)^4 + (\mathcal{C}_{5,n}(\theta))^2} \, . \tag{B.27}$$

From eq. (B.27) one can immediately obtain the hamiltonian of the D-strings. In order to compare this result with the one corresponding to the wrapped D3-brane, let us change the worldsheet coordinate from z to $r = z/\cos\theta$. Recalling that θ is constant for the configurations under study and using that $T_1/\pi = 4\pi\alpha' T_3 = T_3 \Omega_2 \alpha'$, we get the following hamiltonian:

$$H = T_3 \Omega_2 \, N\alpha' \int dr \sqrt{(\sin\theta)^4 + (\mathcal{C}_{5,n}(\theta))^2} \, , \tag{B.28}$$

which, indeed, is the same as in the one in eq. (4.23) for this case. Notice that n, which in our present approach is the number of D-strings, corresponds to the quantization integer of the D3-brane worldvolume gauge field. It follows that the minimal energy configurations occur for $\theta = \pi n/N$ and its energy density is the one written in eq. (4.24). This agreement shows that our ansatz represents D-strings growing up into a D3-brane configuration of the type studied in the main text.

Let us finally point out that the same ansatz of eqs. (B.9) and (B.13) can be used to describe the configurations in which D0-branes expand into a D2-brane in the NS5 background of eqs. (B.3) and (B.4). In this case, which corresponds to the situation analyzed in ref. [1], the D2-branes are located at fixed r and one only has to compute the determinant of the matrix (B.15) in the D0-brane action. By using eq. (B.21) one easily finds the same hamiltonian and minimal energy configurations as those of ref. [1].

References

[1] C. Bachas, M. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[2] J. Pawelczyk, SU(2) *WZW D-branes and their noncommutative geometry from DBI action*, *J. High Energy Phys.* **08** (2000) 006 [hep-th/0003057].

[3] C. Klimcik and P. Severa, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[4] M. Kato and T. Okada, *D-branes on group manifolds*, *Nucl. Phys.* **B 499** (1997) 583 [hep-th/9612148].

[5] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

JHEP05(2001)011

[6] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162–190 [hep-th/9909030];
L. Birke, J. Fuchs and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, *Adv. Theor. Math. Phys.* **3** (1999) 671 [hep-th/9905038].

[7] J. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[8] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017];
see also, J. Polchinski, *Tasi lectures on D-branes*, hep-th/9611050.

[9] K. Gawedzki, *Conformal field theory: a case study*, hep-th/9904145;
S. Stanciu, *D-branes in group manifolds*, *J. High Energy Phys.* **01** (2000) 025 [hep-th/9909163]; *A note on D-branes in group manifolds: flux quantization and d0-charge*, *J. High Energy Phys.* **10** (2000) 015 [hep-th/0006145];
A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040];
Brane dynamics in background fluxes and non-commutative geometry, *J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187];
J.M. Figueroa-O'Farrill and S. Stanciu, *D-brane charge, flux quantization and relative (co)homology*, *J. High Energy Phys.* **01** (2001) 006 [hep-th/0008038];
A. Kling, M. Kreuzer and J.-G. Zhou, SU(2) *WZW D-branes and quantized worldvolume* U(1) *flux on* S^2, *Mod. Phys. Lett.* **A 15** (2000) 2069 [hep-th/0005148];
T. Kubota and J.-G. Zhou, *Rr charges of D2-branes in group manifold and Hanany-Witten effect*, *J. High Energy Phys.* **12** (2000) 030 [hep-th/0010170];
W. Taylor, *D2-branes in B fields*, *J. High Energy Phys.* **07** (2000) 039 [hep-th/0004141];
A. Alekseev, A. Mironov and A. Morozov, *On b-independence of RR charges*, hep-th/0005244;
D. Marolf, *Chern-Simons terms and the three notions of charge*, hep-th/0006117;
A. Alekseev and V. Schomerus, *RR charges of D2-branes in the WZW model*, hep-th/0007096;
J.-G. Zhou, *D-branes in B fields*, hep-th/0102178.

[10] J. Pawelczyk and S.-J. Rey, *Ramond-Ramond flux stabilization of D-branes*, *Phys. Lett.* **B 493** (2000) 395 [hep-th/0007154].

[11] Y. Imamura, *Supersymmetries and BPS configurations on anti-de Sitter space*, *Nucl. Phys.* **B 537** (1999) 184 [hep-th/9807179].

[12] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[13] C.G. Callan Jr., A. Guijosa and K.G. Savvidy, *Baryons and string creation from the fivebrane worldvolume action*, *Nucl. Phys.* **B 547** (1999) 127 [hep-th/9810092];

C.G. Callan Jr., A. Guijosa, K.G. Savvidy and O. Tafjord, *Baryons and flux tubes in confining gauge theories from brane actions*, *Nucl. Phys.* **B 555** (1999) 183 [hep-th/9902197].

[14] B. Craps, J. Gomis, D. Mateos and A.V. Proeyen, *BPS solutions of a D5-brane world volume in a D3-brane background from superalgebras*, *J. High Energy Phys.* **04** (1999) 004 [hep-th/9901060].

[15] J.M. Camino, A.V. Ramallo and J.M. S. de Santos, *Worldvolume dynamics of D-branes in a D-brane background*, *Nucl. Phys.* **B 562** (1999) 103 [hep-th/9905118].

[16] P. Pasti, D. Sorokin and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, *Phys. Lett.* **B 398** (1997) 41 [hep-th/9701037].

[17] M. Alishahiha, $(0,2)$ *theory and string baryon in M-theory on* $AdS_7 \times S^4$, hep-th/9811042.

[18] J. Gomis, A.V. Ramallo, J. Simon and P.K. Townsend, *Supersymmetric baryonic branes*, *J. High Energy Phys.* **11** (1999) 019 [hep-th/9907022].

[19] J.X. Lu and S. Roy, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of type-IIB super five-branes*, *Phys. Lett.* **B 428** (1998) 289 [hep-th/9802080].

[20] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053]; N.R. Constable, R.C. Myers and O. Tafjord, *The noncommutative bion core*, *Phys. Rev.* **D 61** (2000) 106009 [hep-th/9911136].

[21] For a review see, for example, K.S. Stelle, *Lectures on supergravity p-branes*, hep-th/9701088;
R. Argurio, *Brane physics in M-theory*, hep-th/9807171.

[22] G.W. Gibbons and D.A. Rasheed, *Electric - magnetic duality rotations in nonlinear electrodynamics*, *Nucl. Phys.* **B 454** (1995) 185 [hep-th/9506035];
G.W. Gibbons and D.A. Rasheed, $\mathrm{SL}(2,\mathbb{R})$ *invariance of non-linear electrodynamics coupled to an axion and a dilaton*, *Phys. Lett.* **B 365** (1996) 46 [hep-th/9509141];
M.K. Gaillard and B. Zumino, *Self-duality in nonlinear electromagnetism*, hep-th/9705226;
A.A. Tseytlin, *Self-duality of Born-Infeld action and Dirichlet 3-brane of type-IIB superstring theory*, *Nucl. Phys.* **B 469** (1996) 51 [hep-th/9602064];
M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *Dual D-brane actions*, *Nucl. Phys.* **B 496** (1997) 215 [hep-th/9702133].

[23] P.M. Llatas, A.V. Ramallo and J.M. S. de Santos, *World-volume solitons of the D3-brane in the background of (p,q) five-branes*, *Int. J. Mod. Phys.* **A 15** (2000) 4477 [hep-th/9912177].

[24] J. Simon, *Supersymmetric worldvolume solitons*, hep-th/9909005.

[25] O. Pelc, *On the quantization constraints for a d3 brane in the geometry of NS5 branes*, *J. High Energy Phys.* **08** (2000) 030 [hep-th/0007100].

[26] R. Schiappa, *Matrix strings in weakly curved background fields*, hep-th/0005145.

[27] J.M. Maldacena and J.G. Russo, *large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134].

[28] A.A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].